Linear bilevel programming with uncertain lower-level costs

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Abstract

We study a bilevel programming problem with linear data, whose lower-level objective is a random variable with a known distribution. We consider the case where this distribution is nonatomic, allowing to pose the problem of the leader as the optimization of his expected value. We prove that under suitable assumptions this formulation turns out to be piecewise affine over the so-called chamber complex of the feasible region (of the high-point relaxation). We propose two algorithmic approaches to solve general problems enjoying this last property. The first one is based on enumerating the vertices of the chamber complex and evaluating them. The second one is based on the fact that randomly drawn points of the domain should be in the interior of a full-dimensional chamber, where the problem (restricted to this chamber) can be reduced to a linear program.

1 Introduction

Stackelberg games, also referred to as Bilevel programming problems, were first introduced by H. von Stackelberg in [29]. In this seminal work, an economic equilibrium problem between two firms was studied, under the particularity that one of them, the leader, is able to anticipate the decisions of the other one, the follower. This anticipation is possible thanks to the knowledge that the leader has about the follower's decision problem. Bilevel programming is an active field of research, and we refer the reader to the monographs [10, 11] for comprehensive introductions, and to [12] for recent developments.

In the last decade, the assumption of perfect information within Stackelberg games has been questioned. In practice, many problems have the hierarchical leader-follower structure, but only partial information is available for the agents. To deal with these models, researchers have started to consider uncertainty in Stackelberg games. A recent survey by Beck, Ljubić and Schmidt [3] provides an overview of new questions and recent contributions on this topic.

One of the models that consider uncertainty in Stackelberg games is the Bayesian approach [27, 25]. The starting point is that for any given leader decision x, the leader only knows that the reaction y of the follower is selected from a set $S(x)$, hence being y a decision-dependent uncertainty parameter. The leader endows the set $S(x)$ with a probability distribution β_x which models how the leader believes that the possible responses of the follower are distributed.

Uncertainty in the data of the lower-level has been considered by Claus for linear bilevel programming from a variational perspective considering risk measures as well (see the survey [6] with Burtscheidt, and the references therein, and the recent works [7, 8]). In [18], Ivanov considered the cost function of the follower as a bilinear form $\langle (Ax + \xi(\omega)), y\rangle$. Recently, in [5], Buchheim, Henke and Irmai considered a Bilevel version of the continuous Knapsack problem.

In this work, we consider a bilevel programming problem where the lower level objective is uncertain for the leader but follows a prior known distribution. Assuming that this distribution is nonatomic, we show that it induces a weak continuous belief over the feasible reaction map.

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1.1 Problem Formulation

Our study will focus in the setting of linear bilevel programming, which is the case where the data (objective functions and constraints) of the problem is linear. More precisely, we aim to study the problem where the leader decides a vector $x \in \mathbb{R}^{n_x}$ that solves

$$(\mathcal{P}) := \begin{cases} \min\limits_{x} & \langle d_1, x\rangle + \mathbb{E}[\langle d_2, y(x,\omega)\rangle] \\ s.t. & x \in X \\ & y(x,\omega) \text{ solves } \begin{cases} \min\limits_{y} & \langle c(\omega), y\rangle \\ s.t. & y \in Y, \qquad \text{a.s. } \omega \in \Omega, \\ & Ax + By \leq b, \end{cases} \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{m\times n_x}$, $B \in \mathbb{R}^{m\times n_y}$, $b \in \mathbb{R}^m$, $d_1 \in \mathbb{R}^{n_x}$, $d_2 \in \mathbb{R}^{n_y}$, and $c : \Omega \to \mathbb{S}_{n_y}$ is a random vector over a probability space $(\Omega, \Sigma, \mathbb{P})$ with values in the unit sphere of $\mathbb{R}^{n_y}$. The notation carries the usual ambiguity of bilevel problems, which appears whenever the lower level optimal response $y(x,\omega)$ is not uniquely determined for some x. However, we focus our attention here on costs whose distributions are nonatomic (in a sense we will specify later on), which implies that with probability 1, $y(x,\omega)$ is unique for all $x \in X$.

1.2 Our contribution

The goal of this work is to study Problem (1) under the perspective of the Bayesian approach. To be more precise, we aim to understand the underlying distribution of the response $y(x,\omega)$ induced by the cost random vector $c(\omega)$, when the set of admissible responses is given by $S(x) = \{y \in \mathbb{R}^{n_y} : Ax + By \leq b\}$. To do so, we will study how the belief β induced by $c(\cdot)$ in the sense of the Bayesian approach [27] is related to the geometric structure of the set-valued map S. By means of the reformulation given by the Bayesian approach, we find that problem (1) happens to be a piecewise linear optimization problem over what is known as the *chamber complex* of the polytope $D = \{(x,y) : Ax + By \leq b\}$. This concept, well-known in fields like computational geometry and tropical geometry (see, e.g., [13] and the references therein) has not been used (to the best of our knowledge) to analyze Stackelberg problems, until now. However, it is worth mentioning that the geometric structure given by the chamber complex of D was recently applied to the study of stochastic optimization with recourse in [13]. Forcier, Gaubert and Leclère provided,in [13], a polynomial algorithm for the two-stage problem with uncertain recourse cost, where the complexity is studied assuming that the dimension of the recourse decision is fixed. This algorithm is based on the convexity of the cost-to-go function since it relies on a polynomial-time separation scheme for the epigraph of the objective function. As we will show, this cannot be done in Stackelberg games, which illustrates once again the difficulty gap between them and the "easier" two-stage problems.

The contributions of this work can be summarized as follows: First, we reformulate Problem (1) following the Bayesian approach of [27] using a belief β induced by the random cost $c(\omega)$, which we call *vertex-supported belief induced by c*. This is done in Section 2, along with some preliminary notation and some simple facts about the chamber complex associated with Problem (1); Secondly, we show that the objective function of Problem (1) as well as its discretization using Sample Average Approximation methods (see, e.g., [17]) have the property to be *piecewise linear over the chamber complex of D*. That is, we show that for every polytope belonging to the chamber complex of D, the objectives of the aforementioned problems are affine within that polytope. This is done in Section 3.

Section 5 and Section 6 are devoted to propose two methods to solve *piecewise linear problems over the chamber complex of a polytope*. The first proposal, in Section 5, is a deterministic algorithm which is based on a very straight-forward observation: the optimal solution of such a problem must be attained at a vertex of the chamber complex. Thus, we provide a MIP formulation to enumerate all the vertices of the chamber complex. This formulation is fully described, with its advantages, the particularities of

its implementation and its difficulties. In particular, it is directly seen that enumerating the vertices of the chamber complex is at least as hard as enumerating vertices of a polytope. The second proposal, in Section 6, is a Monte-Carlo algorithm which is based on another very simple observation: if a point x is drawn randomly, it will belong to the interior of a full-dimensional chamber with probability 1. Thus, we provide an algorithm that takes a sample of the leader decision, compute the full-dimensional chamber containing the sampled point, and formulates an LP to find the best solution within that chamber. After many repetitions, since the full-dimensional chambers cover the decision space of the leader, the probability of having the optimal solution should be high.
In Section 4 we provide some illustrative examples that allow to understand how hard is to work with the chamber complex of a polytope, and how (a posteriori) the enumeration algorithm and the Monte-Carlo algorithm should compare. We finish with our work by testing both solution methods over some numerical experiments. This is done in Section 7.

2 Preliminaries

This section is devoted to the preliminary concepts that are needed for this work. Here, we provide some key elements of the work, as the announced reformulation of Problem 1, and some properties of the Chamber complex of a polytope.

2.1 General notation

For an integer $n \in \mathbb{N}$, we write $[n] := \{1, \dots, n\}$. Through this work we will consider Euclidean spaces $\mathbb{R}^n$ endowed with their usual norm $\|\cdot\|$ and their inner product $\langle \cdot, \cdot \rangle$. We denote by $\mathbb{B}_n$ and $\mathbb{S}_n$ the unit ball and the unit sphere in $\mathbb{R}^n$, respectively. For a set $A \subset \mathbb{R}^n$, we will write $\operatorname{int}(A)$, $\overline{A}$, $\operatorname{aff}(A)$, to denote its interior, closure and affine hull, respectively. We denote by $\operatorname{ri}(A)$ its relative interior, which is its interior with respect to the affine space $\operatorname{aff}(A)$. We denote the affine dimension of A as $\dim(A)$, which corresponds to the dimension of $\operatorname{aff}(A)$. It is well known that if A is nonempty and convex, then $\operatorname{ri}(A)$ is nonempty and $\overline{\operatorname{ri}(A)} = \overline{A}$. For a function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and for $\alpha \in \mathbb{R}$, we write $[f \leq \alpha]$, $[f \geq \alpha]$ and $[f = \alpha]$ to denote the α-sublevel set, the α-superlevel set, and the α-level set, respectively. As usual, for two vectors $x, y \in \mathbb{R}^n$, we write $x \leq y$ to denote the pointwise inequalities $x_i \leq y_i$, for each $i \in [n]$.
Given a nonempty set $X \subset \mathbb{R}^{n_x}$ and $Y \subset \mathbb{R}^{n_y}$ we write $M : X \rightrightarrows Y$ to denote a set-valued map, that is, a function assigning to each $x \in X$ a (possibly empty) set $M(x)$ in Y. The graph of M is

$$\operatorname{gph} M = \{(x, y) \in X \times Y : y \in M(x)\}.$$

We say that P is a (convex) polyhedron if it can be written as the feasible set of finitely many linear constraints, that is, if $P = \{x \ : \ Ax \leq b\}$ for some matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$. If P is also compact, we say that P is a polytope. We say that a polytope P is full-dimensional if $\dim(P) = n$ (or equivalently, since P is convex, if $\operatorname{int}(P) \neq \emptyset$), and that P is zero-dimensional if $\dim(P) = 0$ (or equivalently if P is a singleton).
A face F of a polyhedron P is the argmax of a linear function over P. For any $k \in \{0, \dots, n\}$ we will write

$$\mathscr{F}_k(P) := \{F \in \mathscr{F}(P) \ : \ \dim(F) = k\}. \tag{2}$$

The zero-dimensional faces are called the vertices of P and we write $\operatorname{ext}(P) := \mathscr{F}_0(P)$.
For a convex set A and a point $x \in A$, we write $N_A(x)$ to denote the usual normal cone of A at x, that is,

$$N_A(x) = \{d \in \mathbb{R}^n \ : \ \langle d, y - x \rangle \leq 0, \forall y \in A\}. \tag{3}$$

2.2 Specific notation

Motivated by the structure of Problem (1), we define the polyhedron D as the feasible region of the High-point relaxation of the problem (see, e.g., [20]), that is,

$$D = \{(x, y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \ : \ Ax + By \leq b\}. \tag{4}$$

It will be assumed along the paper that D is full dimensional. We do not lose generality since it is always possible to embed D into $\mathbb{R}^{\dim(D)}$. We will assume that

$$X = \big\{x \in \mathbb{R}^{n_x} \ : \ \exists y \in \mathbb{R}^{n_y} \text{ such that } Ax + By \leq b\big\}. \tag{5}$$

Similarly, we will denote

$$Y = \big\{y \in \mathbb{R}^{n_y} \ : \ \exists x \in \mathbb{R}^{n_x} \text{ such that } Ax + By \leq b\big\}, \tag{6}$$

and, for simplicity, we will only consider the case where X and Y are compact. We write $\pi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_x}$ to denote the parallel projection given by $\pi(x, y) = x$. In particular, Equation (5) can be written as $X = \pi(D)$. We consider the set-valued map $S : X \rightrightarrows \mathbb{R}^{n_y}$ defined as

$$S(x) = \{y \in \mathbb{R}^{n_y} \ : \ (x, y) \in D\}. \tag{7}$$

We call S the *fiber map* of D (through the projection π). Clearly, S has nonempty convex and compact values, and $S(X) = Y$, as given by equation (6). It is well known that the fiber map S is continuous in the sense of Painlevé-Kuratoski. Moreover, it enjoys a stronger notion of continuity, called *rectangular continuity*, as it is shown [27].
For each $i \in [m]$ we define the function $g_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ given by

$$g_i(x, y) = A_i x + B_i y - b_i, \tag{8}$$

where A_i and B_i denote the ith row of A and B, respectively. Thus, we can write

$$D = \{(x, y) \ : \ g_i(x, y) \leq 0, \forall i \in [m]\}.$$

Since every function g_i is affine, we will simply write ∇g_i to denote their gradients at any point. For any subset $J \subset [m]$, we define the function $g_J : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ given by

$$g_J(x, y) = \sum_{j \in J} g_j(x, y). \tag{9}$$

If there is no ambiguity, we simply write $\mathscr{F} = \mathscr{F}(D)$ and for each $n \in \{0, 1, \ldots, n_x + n_y\}$, $\mathscr{F}_n = \mathscr{F}_n(D)$. Note that, for each face $F \in \mathscr{F}_n$ we can select a set $J =\subset [m]$ of cardinality $|J| = (n_x + n_y) - n$ with

$$F = D \cap [g_J(x, y) = 0]. \tag{10}$$

Thus, for any face $F \in \mathscr{F}(D)$, we define $g_F := g_J$, where J is the (selected) subset of $[m]$ verifying (10). Observe that in such a case, since $|J|$ coincides with the codimension of F, the set $\{\nabla g_j \ : \ j \in J\}$ must be linearly independent.

2.3 Vertex-supported beliefs and Bayesian formulation

In what follows, we write $\mathcal{B}(Y)$ to denote the Borel σ-algebra of Y and $\mathscr{P}(Y)$ to denote the family of all (Borel) probability measures on Y. We endow $\mathscr{P}(Y)$ with the topology of weak convergence. Recall that a sequence $(\nu_n) \subset \mathscr{P}(Y)$ weak converges to $\nu \in \mathscr{P}(Y)$, which is denoted by $\nu_n \xrightarrow{w} \nu$, if

$$\int_Y f(y) d\nu_n(y) \to \int_Y f(y) d\nu(y),$$

for all continuous functions $f : Y \to \mathbb{R}$ with bounded support. It is well known that $\mathscr{P}(Y)$ is metrizable for the induced topology by the weak convergence and, since Y is compact, $\mathscr{P}(Y)$ is compact as well (see, e.g., [21, Chapter 13]). Accordingly, we will say that a measure-valued map $h : X \to \mathscr{P}(Y)$ is weak continuous if for every sequence $(x_n) \in X$ converging to a point $x \in X$, the sequence $(h(x_n)) \subset \mathscr{P}(Y)$ weak converges to $h(x)$.

Recall from [27] that for a set-valued map $M : X \rightrightarrows Y$ with closed nonempty values, a map $\beta : X \to \mathscr{P}(Y)$ is said to be *a belief over* M if for every $x \in X$, the measure $\beta(x) = \beta_x$ concentrates over $M(x)$, that is, $\beta_x(M(x)) = 1$. By identifying $\mathscr{P}(M(x))$ with its natural injection into $\mathscr{P}(Y)$, a belief over M can be understood as a selection of $\{\mathscr{P}(M(x)) : x \in X\}$.

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. To model the cost function of the lower-level problem in (1), we will consider only random vectors $c : \Omega \to \mathbb{S}_{n_y}$ with nonatomic distributions, in the sense that

$$\forall O \in \mathcal{B}(\mathbb{S}_{n_y}), \mathcal{H}^{n_y-1}(O) = 0 \implies \mathbb{P}[c(\omega) \in O] = 0, \tag{11}$$

where $\mathcal{B}(\mathbb{S}_{n_y})$ stands for the Borel σ-algebra of $\mathbb{S}_{n_y}$, and $\mathcal{H}^{n_x-1}$ denotes the (n_y-1)-Hausdorff measure over the $(\mathbb{S}_{n_x}, \mathcal{B}(\mathbb{S}_{n_y}))$. In other words, the probability measure $\mathbb{P} \circ c^{-1}$ is absolutely continuous with respect to $\mathcal{H}^{n_y-1}$. Note that with this definition, any random vector $c : \Omega \to \mathbb{R}^{n_y}$ which has an absolutely continuous distribution with respect to the Lebesgue measure $\mathcal{L}^{n_y}$ induces an equivalent random vector $\bar{c} : \Omega \to \mathbb{S}_{n_y}$ given by $\bar{c}(\omega) = \frac{c(\omega)}{\|c(\omega)\|}$. This new random vector is well-defined almost surely in Ω, excepting for the negligible set $N = c^{-1}(0)$, and using $c(\cdot)$ or $\bar{c}(\cdot)$ in Problem (1) is equivalent.

Now, to understand the distribution of the optimal response $y(x, \omega)$ induced by the random $c : \Omega \to \mathbb{S}_{n_y}$, we consider a belief $\beta : X \to \mathscr{P}(Y)$ over the fiber map $S : X \rightrightarrows Y$ given by

$$d\beta_x(y) = p_c(x, y) := \mathbb{P}[-c(\omega) \in N_{S(x)}(y)]. \tag{12}$$

Note that $\mathbb{P}[-c(\omega) \in N_{S(x)}(y)] = \mathbb{P}[-c(\omega) \in \operatorname{int}(N_{S(x)}(y)) \cap \mathbb{S}_{n_y}]$ for any $(x, y) \in D$, and that $\operatorname{int}(N_{S(x)}(y))$ is nonempty only if y is an extreme point of $S(x)$. By mixing both observations, one can easily deduce for each $x \in X$, the function $p(x, \cdot)$ is a discrete density with support contained in $\operatorname{ext}(S(x))$. Therefore, the belief β is given by

$$\beta_x(O) = \sum_{y \in \operatorname{ext}(S(x))} p_c(x, y) \mathbb{1}_O(y), \quad \forall O \in \mathcal{B}(Y). \tag{13}$$

We call β the *vertex-supported belief induced by* c. With this construction, we can rewrite Problem (1) as

$$(\mathcal{P}) := \begin{cases} \min_x & \langle d_1, x \rangle + \mathbb{E}_{\beta_x}[\langle d_2, y \rangle] \\ \\ s.t. & x \in X, \end{cases} \tag{14}$$

where $\mathbb{E}_{\beta_x}[\langle d_2, y \rangle] = \sum_{y \in \operatorname{ext}(S(x))} \langle d_2, y \rangle p_c(x, y)$. Our goal in this work is to study Problem (1) by profiting from the Bayesian formulation (14), in the sense of [27].

Remark 2.1. *Note that, by defining the centroid of $S(x)$ with respect to β_x as*

$$\mathfrak{b}(x) := \mathbb{E}_{\beta_x}[y] = \sum_{y \in \operatorname{ext}(S(x))} p_c(x, y) y, \tag{15}$$

we get that $\theta(x) = \langle d_1, x \rangle + \langle d_2, \mathfrak{b}(x) \rangle$. Thus, the Bayesian formulation (14) can be rewritten in terms of the centroid, which is the convex combination of the vertices of $S(x)$, proportionally to the weight of their normal cones. Of course, this is only valid since the expected value $\mathbb{E}_{\beta_x}[\langle d_2, y \rangle]$ has a linear integrand.

Remark 2.2. *When the distribution of $c(\cdot)$ is atomic (i.e. it assigns positive probability to negligible sets of the unit sphere $\mathbb{S}_{n_y}$), then one must deal with the fact that $y(x, \omega)$ is not necessarily uniquely defined. In this case, the construction of the density $p_c(x, \cdot)$ is not obvious, and therefore it is not clear how to construct an appropriated vertex-supported belief induced by $c(\cdot)$.*

2.4 The Chamber complex

We start by recalling the definitions of polyhedral complex and chamber complex, well-known in some fields of mathematics, like computational geometry (see, e.g., [9]).

Definition 2.3 (Polyhedral complex)**.** *A set of polyhedra $\mathscr{P}$ in $\mathbb{R}^{n_x}$ is a polyhedral complex if*

1. *For each $P, Q \in \mathscr{P}$, $P \cap Q$ is a face of Q and of P*
2. *For each $P \in \mathscr{P}$ and each face F of P, $F \in \mathscr{P}$.*

We refer to the elements of $\mathscr{P}$ as cells.

Given a cell $P \in \mathscr{P}$, each vertex of P (as a face of P) belongs to $\mathscr{P}$ and moreover these are the minimal elements of $\mathscr{P}$. If the maximal cells in $\mathscr{P}$ have the same dimension, we say that $\mathscr{P}$ is a pure polyhedral complex. We will be interested in a particular class of polyhedral complex that are pure, the so called Chamber complex, which we recall next.

Definition 2.4 (Chamber complex)**.** *Let $D \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ be a polyhedron as described in* (4)*. For each $x \in X = \pi(D)$, we define the chamber of x as*

$$\sigma(x) = \bigcap \big\{\pi(F) \,:\, F \in \mathscr{F}(D),\, x \in \pi(F)\big\}. \tag{16}$$

*The **Chamber complex**, is given then by the (finite) collection of chambers, that is,*

$$\mathscr{C}(D) = \{\sigma(x) \,:\, x \in X\}. \tag{17}$$

In what follows, it will be particularly useful to distinguish the maximal chambers, which are the chambers with nonempty interior, and the minimal chambers, which are the vertices in the chamber complex. The following definition introduces such distinction.

Definition 2.5. *Let $D \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ be a polyhedron as described in* (4)*. We define the families*

$$\mathscr{K}(D) := \{K \in \mathscr{C}(D) \;:\; \operatorname{int}(K) \neq \emptyset\} \tag{18}$$

$$\mathscr{V}(D) := \{v \in X \;:\; \{v\} \in \mathscr{C}(D)\} \tag{19}$$

We call $\mathscr{K}(D)$ the family of full-dimensional chambers of D, and $\mathscr{V}(D)$, the vertices of the chamber complex $\mathscr{C}(D)$ (which correspond to the family of zero-dimensional chambers of D).

It is worth to mention that $\mathscr{K}(D)$ is a covering of X (more generally, the maximal cells of a polyhedral complex cover the union of all cells). This follows from the fact that each chamber is contained in a full-dimensional chamber.
The next proposition gives us some facts about the chamber of a point, and how one could compute it.

Proposition 2.6. *For any $\bar{x} \in X$, one has that*

1. $\sigma(\bar{x}) = \bigcap\{\pi(F) \;:\; F \in \mathscr{F},\, \bar{x} \in \pi(F),\, \dim(F) \leq n_x\}$.
2. $\bar{x} \in \operatorname{ri}(\sigma(\bar{x}))$.

Furthermore, for any chamber $C \in \mathscr{C}(D)$ and every $x \in \operatorname{ri}(C)$, one has that $C = \sigma(x)$.

Proof. For the first assertion, let us consider $F \in \mathscr{F}$ with $\bar{x} \in \pi(F)$ such that $\dim(F) = n > n_x$. Then, there exists $J \subset [m]$ with $|J| = (n_x + n_y) - n < n_y$ such that $F = D \cap [g_J(x, y) = 0]$. Now, choose $\bar{y} \in S(\bar{x})$ such that the point $(\bar{x}, \bar{y}) \in F$ is a vertex of $\{\bar{x}\} \cap S(\bar{x})$. Let $\hat{J} \subset [m]$ the set of all active constraints at $(\bar{x}, \bar{y})$. Then, $J \subset \hat{J}$ and $\{(\bar{x}, \bar{y})\} = [g_{\hat{J}}(x, y) = 0, x = \bar{x}]$. This implies that

$$\{\nabla g_j \;:\; j \in \hat{J}\} \cup \{(e_k, 0) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \;:\; k \in [n_x]\},$$

has n_x+n_y linearly independent vectors. In particular, $\{\nabla g_j \ : \ j \in \hat{J}\}$ has at least n_y linearly independent vectors, which yields that $\hat{F} = D \cap [g_{\hat{J}}(x,y) = 0]$ satisfies that $\dim(\hat{F}) \leq n_x$. Then, since $J \subset \hat{J}$, we have that $\hat{F} \subset F$, which yields that $\pi(\hat{F}) \subset \pi(F)$. The arbitrariness of F allows us to write

$$\begin{aligned}\sigma(\bar{x}) &= \bigcap \{\pi(F) \ : \ F \in \mathscr{F},\, \bar{x} \in \pi(F)\} \\ &\supset \bigcap \{\pi(F) \ : \ F \in \mathscr{F},\, \bar{x} \in \pi(F),\, \dim(F) \leq n_x\},\end{aligned}$$

finishing the proof of the first assertion.
For the second part, since $\{\mathrm{ri}(C) \ : \ C \in \mathscr{C}(D)\}$ is a partition of X, we know there exists $z \in X$ such that $\bar{x} \in \mathrm{ri}(\sigma(z))$. Then $\sigma(\bar{x}) \subset \sigma(z)$ and so $\sigma(\bar{x})$ is a proper face of $\sigma(z)$, so that $\sigma(\bar{x}) \cap \mathrm{ri}(\sigma(z)) = \emptyset$. But this implies that $\bar{x} \notin \sigma(\bar{x})$ which is evidently a contradiction.
For the last part, fix $C \in \mathscr{C}$ and let $x \in \mathrm{ri}(C)$. Since $x \in \mathrm{ri}(\sigma(x))$ we get that $\mathrm{ri}(\sigma(x)) \cap \mathrm{ri}(C) \neq \emptyset$, which finishes the proof by recalling again that $\{\mathrm{ri}(C') \ : \ C' \in \mathscr{C}(D)\}$ is a partition of X.

□

We finish this preliminary section with the following direct proposition that relates the minimal and maximal chambers, in the sense that the vertices of the chamber complex are the collection of the vertices of all full-dimensional chambers.

Proposition 2.7. *Given D as in (4) with nonempty interior then*

$$\mathscr{V}(D) = \bigcup_{K \in \mathscr{K}(D)} \mathrm{ext}(K).$$

Proof. Since $\mathscr{C}(D)$ is a polyhedral complex (see, e.g., [9]), a vertex of a given chamber is itself a chamber, and hence an element of $\mathscr{V}(D)$. This proves the inclusion to the left.
For the inclusion to the right take $v \in \mathscr{V}(D)$. Since $X = \bigcup_{K \in \mathscr{K}(D)} K$, then $\{v\}$ is a chamber that is included in some maximal chamber $K \in \mathscr{K}(D)$. But then $\{v\}$ is a vertex of K again from the property of polyhedral complexes. □

3 Geometrical structure of vertex-supported belief

From the properties of the chamber complex it is easy to deduce that continuous piecewise linear functions attain their extreme values over $\mathscr{V}(D)$. Definition 3.1 and Corollary 3.2 formalize this intuition.

Definition 3.1. *A function $f : X \to \mathbb{R}$ is said to be* piecewise linear over the chamber complex $\mathscr{C}(D)$ *if there exists a sequence of pairs $\{(d_K, a_k) \ : \ K \in \mathscr{C}(D)\} \subset \mathbb{R}^{n_x} \times \mathbb{R}$ such that*

$$f(x) = \sum_{K \in \mathscr{C}(D)} (\langle d_K, x \rangle + a_K) \mathbb{1}_{\mathrm{ri}(K)}(x), \qquad \forall x \in X. \tag{20}$$

Corollary 3.2. *If a function $f : X \to \mathbb{R}$ is continuous and piecewise linear over the chamber complex $\mathscr{C}(D)$, then it has at least one minimizer in $\mathscr{V}(D)$.*

Proof. Since f is continuous and X is nonempty and compact, Weierstrass theorem entails that f attains its minimum at some point $x^* \in X$. Then, there exist a chamber $K \in \mathscr{C}(D)$ such that $x^* \in K$. Since f is piecewise linear over the chamber complex and continuous, we have that $f\big|_K$ is affine. Thus, there exists $v^* \in \mathrm{ext}(K)$ such that $f(v^*) = f\big|_K(v^*) \leq f\big|_K(x^*) = f(x^*)$. Proposition 2.7 yields that $v^* \in \mathscr{V}(D)$, which finishes the proof. □

3.1 Piecewise affinity of the fiber map

We show in this section that the fiber map S defined in is piecewise affine, in the sense of the following definition. Some properties in this subsection seem to be known, but are hard to find in the literature in the present form. Here, we provide a presentation (and proofs) that are exclusively based on mathematical programming tools. Here, we also present one of our main contributions, which is Theorem 3.6.

Definition 3.3. *A set-valued map $M : X \subset \mathbb{R}^{n_x} \rightrightarrows \mathbb{R}^{n_y}$ is* affine *on a convex set $K \subset \mathrm{dom}(M)$ if*

$$M(\eta x_1 + (1-\eta)x_0) = \eta M(x_1) + (1-\eta)M(x_0),$$

for all $x_0, x_1 \in K$ and $\eta \in (0,1)$, where the addition of sets is understood in the sense of Minkowski. We say that M is piecewise affine*, if the domain of M can be written as a finite union of convex subsets where, in each one of them, the set-valued map is affine.*

Note that if a set-valued map M is affine over K and there exists $\bar{x} \in \mathrm{ri}(K)$ such that $|M(\bar{x})| = 1$, then M must be single-valued over K. Indeed, consider $x \in K$ and consider arbitrary points $y', y'' \in M(x)$. Since $\bar{x} \in \mathrm{ri}(K)$ we can find $\tilde{x} \in K$ and $\eta \in (0,1)$ such that $\bar{x} = \eta\tilde{x} + (1-\eta)x$. We can take $\tilde{y} \in M(\tilde{x})$. But then both $\bar{y}' = \eta\tilde{y} + (1-\eta)y'$ and $\bar{y}'' = \eta\tilde{y} + (1-\eta)y''$ belong to $S(\bar{x})$ which implies that $\bar{y}' = \bar{y}''$ and this yields that $y' = y''$. So $M(x)$ contains a single point, and from the arbitrariness of x, we conclude that S is single-valued in K. With this observation, we can establish the following lemma.

Lemma 3.4. *Let $M : X \subset \mathbb{R}^{n_x} \rightrightarrows \mathbb{R}^{n_y}$ be affine on a convex set $K \subset \mathrm{dom}(M)$, and consider a linear functional $\psi : \mathbb{R}^{n_y} \to \mathbb{R}$ such that for some $x \in K$, $\sup \psi(M(x))$ is finite and attained.*

1. *The function $\varphi(x) := \sup \psi(M(x))$ is affine.*
2. *The set-valued map $M_\psi(x) := \mathrm{argmax}_{y \in M(x)} \psi(y)$ is affine on K. Additionally, if there exists a point $\bar{x} \in \mathrm{ri}(K)$, with $|M_\psi(\bar{x})| = 1$, then M_ψ is single-valued on K.*

Proof. Fix $x_0, x_1 \in K$ and $\eta \in (0,1)$, and let $x_\eta = \eta x_1 + (1-\eta)x_0$. The first part of the Lemma follows from

$$\begin{aligned}
\sup \psi(M(x_\eta)) &= \sup \psi(\eta M(x_1) + (1-\eta)M(x_0)) \\
&= \sup\{\eta\psi(M(x_1)) + (1-\eta)\psi(M(x_0))\} \\
&= \sup\{\eta\psi(M(x_1))\} + \sup\{(1-\eta)\psi(M(x_0))\} \\
&= \eta\varphi(x_1) + (1-\eta)\varphi(x_0).
\end{aligned}$$

Let us now prove the second part. Indeed, if $y \in M_\psi(x_\eta)$, we have that $y \in M(x_\eta)$ and since M is affine, then $y = \eta y_1 + (1-\eta)y_0$ for some $y_0 \in M(x_0)$ and $y_1 \in M(x_1)$. Since φ is affine we have that

$$\begin{aligned}
\varphi(x_\eta) &= \eta\varphi(x_1) + (1-\eta)\varphi(x_0) \\
&\geq \eta\psi(y_1) + (1-\eta)\psi(y_0) \\
&= \psi(y) \\
&= \varphi(x_\eta).
\end{aligned}$$

But then $\eta\varphi(x_1) + (1-\eta)\varphi(x_0) = \eta\psi(y_1) + (1-\eta)\psi(y_0)$ and this can only occur if $\psi(y_1) = \varphi(x_1)$ and $\psi(y_0) = \varphi(x_0)$. Thus we have $y_1 \in M_\psi(x_1)$ and $y_0 \in M_\psi(x_0)$. We have hence proved the inclusion

$$M_\psi(x_\eta) \subset \eta M(x_1) + (1-\eta)M(x_0).$$

The reverse inclusion corresponds to the convexity of the graph of M_ψ, which follows directly from the equality

$$\mathrm{gph}\, M_\psi = \mathrm{gph}\, M \cap \{(x,y) : \psi(y) - \varphi(x) \geq 0\}.$$

□

Lemma 3.5. *The fiber map S is piecewise affine over the chamber complex $\mathscr{C}(D)$. More precisely, for any chamber $K \subset \mathscr{C}(D)$, S is affine within K.*

Proof. Let $K \in \mathscr{C}(D)$ and let $v_1, ..., v_k$ be the vertices of K. If $x_0, x_1 \in K$ then

$$x_0 = \sum_{i=1}^{k} \lambda_{0,i} v_i \quad x_1 = \sum_{i=1}^{k} \lambda_{1,i} v_i$$

for some $\lambda_{0,i}, \lambda_{1,i} \geq 0$, $i = 1, ..., k$, such that $\sum_{i=1}^{k} \lambda_{0,i} = \sum_{i=1}^{k} \lambda_{1,i} = 1$. Consider now a point $x_2 = \eta x_1 + (1-\eta)x_0$ for some $\eta \in (0,1)$. We clearly have that $x_2 = \sum_{i=1}^{k} \lambda_{2,i} v_i$ with $\lambda_{2,i} = \eta\lambda_{0,1} + (1-\eta)\lambda_{0,i}$, $i = 1, ..., k$.
Applying the representation of [26, Proposition 2.4], we know that for each convex combination $x = \sum_{i=1}^{k} \lambda_i v_i \in K$, we have that $S(x) = \sum_{i=1}^{k} \lambda_i S(v_i)$. Thus, we can write

$$\begin{aligned} S(x_2) = \sum_{i=1}^{k} \lambda_{2,i} S(v_i) &= \eta \sum_{i=1}^{k} \lambda_{1,i} S(v_i) + (1-\eta) \sum_{i=1}^{k} \lambda_{0,i} S(v_i) \\ &= \eta S(x_1) + (1-\eta) S(x_0). \end{aligned}$$

The proof is then finished. □

Theorem 3.6. *Consider a random cost c with nonatomic distribution and the vertex-supported belief $\beta : X \to \mathscr{P}(Y)$ over S induced by c as defined in* (13)*. Then,*

1. *β is weak-continuous, and thus for any lower semicontinuous function $f : X \times Y \to \mathbb{R}$, the problem $\min_{x \in X} \mathbb{E}_{\beta_x}[f(x, \cdot)]$ has a solution.*
2. *The function $\theta : X \to \mathbb{R}$ given by $\theta(x) = \langle d_1, x \rangle + \mathbb{E}_{\beta_x}[\langle d_2, y \rangle]$ is continuous and piecewise linear over $\mathscr{C}(D)$.*

In particular, Problem (1) *has at least one solution over $\mathscr{V}(D)$.*

Proof. Since $\mathscr{K}(D)$ is a covering of X, then it is enough to prove the assertion within a given chamber $K \in \mathscr{K}(D)$. Since S is affine on K with compact polyhedral images, we first claim that there exists affine functions $y_1(x), y_2(x), ..., y_k(x)$ such that for all $x \in K$ one has that $\{y_1(x), y_2(x), ..., y_k(x)\} = \text{ext}(S(x))$ or, equivalently, that

$$\{y_1(x), y_2(x), ..., y_k(x)\} \subset \text{ext}(S(x)) \quad \text{and} \quad S(x) = \text{conv}(y_1(x), y_2(x), ..., y_k(x)).$$

Indeed, let $\bar{x} \in \text{int}\, K$ and consider the vertices $\bar{y}_1, ..., \bar{y}_k$ of $S(\bar{x})$. For each i there exists a linear functional ψ_i that exposes $\bar{y}_i$, that is, such that $\{\bar{y}_i\} = \{y \in S(\bar{x}) : \psi_i(y) \geq \psi_i(\bar{y}_i)\}$. Then from Lemma 3.4 the maximum of ψ_i over $S(x)$ defines the affine function

$$x \mapsto y_i(x) = \text{argmax}\{\psi_i(y) : y \in S(x)\},$$

for each $i \in [k]$. This proves that $\{y_1(x), y_2(x), ..., y_k(x)\} \subset \text{ext}(S(x))$ for each $x \in K$. Now, it is direct that $S(x) \supseteq \text{conv}(y_1(x), y_2(x), ..., y_k(x))$, for each $x \in K$, and so we only need to prove the direct inclusion. Suppose that this is not the case and choose $x \in K$ and $y \in S(x) \setminus M$ with $M = \text{conv}(y_1(x), y_2(x), ..., y_k(x))$. Without losing any generality, there exist a linear functional ψ that separates y from M, and so $\sup \psi(S(x)) > \sup \psi(M)$. Now, since $\bar{x} \in \text{int}(K)$, there exists $x' \in K$ and $\eta \in (0,1)$ such that $\bar{x} = \eta x + (1-\eta)x'$. Let $M' = \text{conv}(y_1(x'), y_2(x'), ..., y_k(x'))$. Then, by affinity of the functions $y_1, \ldots, y_k$, one has that $S(\bar{x}) = \eta M + (1-\eta)M'$. Then, by Lemmas 3.4 and 3.5 we can write

$$\begin{aligned} \sup \psi(S(\bar{x})) &= \sup \psi(\eta M + (1-\eta)M') \\ &= \eta \sup \psi(M) + (1-\eta) \sup \psi(M') \\ &< \eta \sup \psi(S(x)) + (1-\eta) \sup \psi(S(x')) = \sup \psi(S(\bar{x})), \end{aligned}$$

which is a contradiction. This proves our first claim. Now, we claim that for each $i \in [k]$

$$N_{S(x)}(y_i(x)) = \bigcup_{j\in[k]:y_j(x)=y_i(x)} N_{S(\bar{x})}(\bar{y}_j), \quad \forall x \in K. \tag{21}$$

Indeed, for the direct inclusion, fix $x \in K$ and choose $\psi \in \text{int}(N_{S(x)}(y_i(x)))$, which is nonempty since $y_i(x) \in \text{ext}(S(x))$. This means that ψ attains its maximum on $S(x)$ only at $y_i(x)$. Now, ψ attains its maximum over $S(\bar{x})$ at some $\bar{y}_j$, and so $\psi \in N_{S(\bar{x})}(\bar{y}_j)$. Since $\bar{x} \in \text{int}(K)$, there exists $x' \in K$ and $\eta \in (0,1)$ such that $\bar{x} = \eta x + (1-\eta)x'$. Then, by construction, $\bar{y}_j = \eta y_j(x) + (1-\eta)y_j(x')$, which yields that ψ attains its maximum on $S(x)$ at $y_j(x)$. This yields that $y_i(x) = y_j(x)$, and so $\psi \in \bigcup_{j\in[k]:y_j(x)=y_i(x)} N_{S(\bar{x})}(\bar{y}_j)$. The inclusion is proven by taking closure of $\text{int}(N_{S(x)}(y_i(x)))$. Now, for the reverse inclusion, choose $j \in [k]$ such that $y_i(x) = y_j(x)$ and $\psi \in N_{S(\bar{x})}(\bar{y}_j)$. Then, ψ attains its maximum on $S(\bar{x})$ at $\bar{y}_j$ and so, replicating the argument above, we have that ψ attains its maximum on $S(x)$ at $y_j(x)$. Since $y_i(x) = y_j(x)$, the conclusion follows. This proves the second claim.
Now, (21) entails that $x \mapsto N_{S(x)}(y_i(x))$ is constant in $\text{int}(K)$ since for every $x \in \text{int}(K)$, one has that $y_i(x) \neq y_j(x)$ if and only if $i \neq j$. Then for each $i \in [k]$ it is clear that

$$p_c(x, y_i(x)) = \mathbb{P}[-c(\omega) \in N_{S(x)}(y_i(x))]$$

is constant in $\text{int}(K)$, equal to some real c_i, and for points x in the boundary of K it holds $p_c(x, y_i(x)) = \sum\{c_j : j \in [k],\, y_i(x) = y_j(x)\}$. Thus, it follows that for any continuous function $h : Y \to \mathbb{R}$ we have

$$\mathbb{E}_{\beta x}[h] = \sum_{i=1}^{k} h(y_i(x))c_i, \quad \forall x \in K,$$

which is continuous on K. Then, $x \mapsto \mathbb{E}_{\beta x}[h]$ must be continuous on X and, since h is arbitrary, a mild application of Portemanteau's theorem (see, e.g., [21, Theorem 13.16]) β is a weak continuous belief. Then, for every lower semicontinuous function $f : X \times Y \to \mathbb{R}$, $\mathbb{E}_{\beta_x}[f(x,\cdot)]$ is lower semicontinuous as well (see [27]). Finally, from equation (3.1) it is clear that taking $f(x,y) = \langle d_1, x\rangle + \langle d_2, y\rangle$, the function $\theta(x) = \mathbb{E}_{\beta_x}[f(x,\cdot)]$ is affine over K. □

3.2 Sample Average Formulation

Problem (1) has an intrinsic difficulty which consist in how to evaluate the objective function $\theta(x) = \langle d_1, x\rangle + \mathbb{E}_{\beta_x}[\langle d_2, y\rangle]$. To make an exact evaluation of θ at a point $x \in X$ one would require to compute the set of all vertices $y_1, ..., y_k$ of $S(x)$ and to compute the values $c_1, ..., c_k$ defined as the "sizes" of the respective normal cones at each vertex y_i. This is not always possible.

To deal with this issue, we consider the well-known Sample Average Approximation (SAA) method for stochastic optimization (see, e.g., [28, 17]). That is, we take an independent sample $\{\hat{c}_1, \ldots, \hat{c}_{N_0}\}$ of the random lower-level cost $c(\cdot)$, where each sample is drawn following the (known) distribution of $c(\cdot)$, and try to solve the (now deterministic) problem

$$(\hat{\mathcal{P}}) := \left\{ \begin{array}{ll} \min_x & \langle d_1, x\rangle + \frac{1}{N_0}\sum_{i=1}^{N_0}\langle d_2, y_i(x)\rangle \\ s.t. & x \in X \\ & \forall i \in [N_0],\, y_i(x) \text{ solves } \left\{ \begin{array}{ll} \min_y & \langle \hat{c}_i, y\rangle \\ s.t. & Ax + By \leq b. \end{array} \right. \end{array} \right. \tag{22}$$

Proposition 3.7. *Assume that $c(\cdot)$ has a nonatomic distribution over $\mathbb{S}_{n_y}$, in the sense of* (11)*. Then, with probability 1, we have that*

$$\text{argmin}_y \{\langle \hat{c}_i, y\rangle \;:\; Ax + By \leq b\} \text{ is a singleton}, \quad \forall i \in [N_0], \forall x \in X.$$

Proof. Consider the set

$$\mathcal{N} = \left\{\omega \in \Omega \,:\, \exists x \in X \text{ such that } \exists y_1 \neq y_2 \in \operatorname{argmin}_y \left\{\langle \hat{c}_i(\omega), y\rangle : Ax + By \leq b\right\}\right\}.$$

Let $\omega \in \mathcal{N}$, and choose $x \in X$ and $y_1, y_2 \in \operatorname{argmin}_y \{\langle \hat{c}_i(\omega), y\rangle : Ax + By \leq b\}$ as in the definition of $\mathcal{N}$. We claim that $\hat{c}_i(\omega)$ must be a linear combination of no more that $n_y - 1$ columns of B. If such a claim holds true, then for any subset $\{b_1, \ldots, b_r\}$ of columns of B^T with $r \leq n_y - 1$ one has that

$$\hat{c}_i \text{ is a linear combination of } \{b_1, \ldots, b_r\} \iff \hat{c} \in \underbrace{\mathbb{S}_{n_y} \cap \operatorname{span}\{b_1, \ldots, b_r\}}_{0:H(b_1,\ldots,b_r)}.$$

Thus, noting that each set $H(b_1, \ldots, b_r)$ is a subset of $\mathbb{S}_{n_y}$ with empty interior relative to $\mathbb{S}_{n_y}$, it must have null measure with respect to the Hausdorff measure $\mathcal{H}^{n_y-1}$. The conclusion follows by noting that there are finitely many such sets $\{b_1, \ldots, b_r\}$.
Let us prove the claim. Indeed, by optimality conditions there exists $\mu_i \geq 0$ and $j_i \in \{1, ..., m\}$ $i = 1, ..., r$ such that

$$0 = \hat{c}_i(\omega) + \sum_{i=1}^{r} \mu_i (B_{j_i})^T \quad \text{and} \quad \mu_i(B_{j_i} y + A_{j_i} x - b_i) = 0$$

for some $r \in [m]$, valid for $y = y_1$ and for $y = y_2$. From the Carathéodory Theorem [2] we can assume without loss of generality that $r = n_y$. If at least one μ_i is equal to zero the proof is done. Otherwise, if all μ_i are nonzero then it must hold

$$B_{j_i} y + A_{j_i} x - b_i = 0$$

so that $B_{j_i}(y_1 - y_2) = 0$. Hence in the sum $\sum_{i=1}^{r} \mu_i (B_{j_i})^T$ at least one vector is a linear combination of the rest and we can write c as a linear combination of $n_y - 1$ columns of B^T.

□

Based on the above proposition, for the sample $\{\hat{c}_1, \ldots, \hat{c}_{N_0}\}$ we can, for each $i \in [N_0]$, define the mapping $x \mapsto y_i(x)$ where $y_i(x)$ is the unique element $\operatorname{argmin}_y \{\langle \hat{c}_i, y\rangle : Ax + By \leq b\}$. The following direct theorem shows that the Sample Average Approximation maintains the main property that we will use in the sequel: the objective function is, almost surely, piecewise linear over the chamber complex $\mathscr{C}(D)$.

Theorem 3.8. *If c has a nonatomic distribution over $\mathbb{S}_{n_y}$, then, with probability 1, the function $\theta : X \to \mathbb{R}$ given by $\theta(x) = \langle d_1, x\rangle + \frac{1}{N}\sum_{i=1}^{N} \langle d_2, y_i(x)\rangle$ is well-defined and it is piecewise linear over the Chamber complex $\mathscr{C}(D)$.*

Proof. With probability 1, the functions $x \mapsto y_i(x)$ are single-valued by Proposition 3.7. Moreover, they are affine over $\mathscr{C}(D)$ thanks to Lemma 3.4. Hence, θ is clearly piecewise affine over $\mathscr{C}(D)$. □

We finish this section by showing that Problem (22) is a consistent approximation of the original Problem (14).

Proposition 3.9. *Let x^* be an optimal solution of Problem* (22) *for a given sample of size N_0, and let ν^* be the associated optimal value. Let $\mathcal{S}$ be the set of solutions of Problem* (1) *and let $\bar{\nu}$ be its optimal value. Then,*

1. *$d(x^*, \mathcal{S}) \xrightarrow{N_0 \to \infty} 0$, w.p.1.*
2. *$\nu^* \xrightarrow{N_0 \to \infty} \bar{\nu}$, w.p.1.*

Proof. First we observe, that with probability 1, $f_{N_0}(x) := \sum_{i=1}^{N_0} \langle d_2, y_i(x)\rangle$ converges to the function $f(x) := \mathbb{E}[\langle d_2, y_i(x)\rangle]$ pointwisely as $N_0 \to \infty$. Then, since $\mathscr{V}(D)$ is finite, we have that, with probability 1, f_{N_0} converges to f uniformly on $\mathscr{V}(D)$. That is, for almost every infinite sample $\{\hat{c}_i(\omega)\}_{i\in\mathbb{N}}$, one has that $\sup_{x\in\mathscr{V}(D)} |f_{N_0}(x) - f(x)| \to 0$ as $N_0 \to \infty$.

Consider a full-dimensional chamber $K \in \mathscr{K}(D)$. Then affinity of f_{N_0} and f on K yields that

$$\sup_{x\in K} |f_{N_0}(x) - f(x)| = \sup_{x\in \text{ext}(K)} |f_{N_0}(x) - f(x)| \leq \sup_{x\in \mathscr{V}(D)} |f_{N_0}(x) - f(x)|,$$

where the last inequality follows from the inclusion $\text{ext}(K) \subset \mathscr{V}(D)$ (see Proposition 2.7). Thus, with probability 1, the convergence is uniform on K. Since there are finitely many full-dimensional chambers and they cover X, we conclude that the convergence is, with probability 1, uniform in X as well. Thus from [28, Theorem 5.3] we conclude the desired properties. □

To asses how good might be the solution x^* of Problem (22), in the sense of how small is the optimality gap $\nu^* - \bar{\nu}$, we follow a standard method called *Multiple Replication Procedure* (MRP), introduced in [24]. Basically, this method allows us to provide a confidence interval for the optimality gap associated to $x^*_{N_0}$ by solving Problem (22) several times with different samplings. We refer the reader to [17, 24] for the details.

4 Some illustrative examples

This section is devoted to providing some examples illustrating key ideas and difficulties of the chamber complex. We also revise in detail the example of Bilevel continuous Knapsack problem with uncertain costs, which has been recently studied in [5].

Example 4.1. Consider $D \in \mathbb{R}^2 \times \mathbb{R}$ the polytope defined as tuples (x_1, x_2, y) such that

$$x_1 - y \leq 0,\ -x_1 - y \leq 0,\ x_2 - y \leq 1,\ \text{and}\ \ -x_2 + y \leq 1.$$

Clearly the vertices of D are $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 1)$, $v_3 = (-1, 0, 0)$ and $v_4 = (0, -1, 1)$ (see Figure 1). We observe that $(0, 0)$ is a vertex of the chamber complex $C(D)$ while it cannot be obtained as the projection of any of the vertices of D.

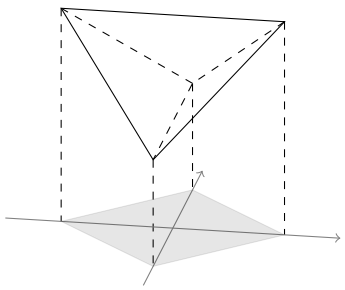


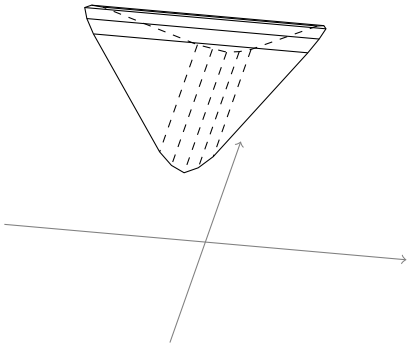
Figure 1: The chamber vertex $(0, 0)$ of $\mathscr{V}(D)$ is not a projection of $\text{ext}(D)$.

Example 4.2 ($\mathscr{V}(D)$ can be arbitrarily large)**.** Let $n \in \mathbb{N}$ and fix $\varepsilon \in (0, 1)$. For each $i = 0, ..., n$ define $\alpha_i = \frac{\varepsilon i}{2n^2}(2n - i + 1)$ and $\beta_i = 1 - \frac{i}{n}$. Now consider the following polytope

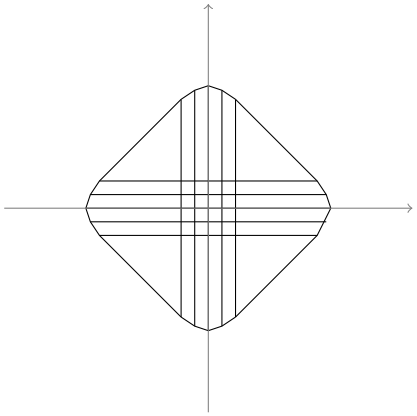
$$D_n = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} : \alpha_i + \beta_i|x_1| \leq y \leq 1 - \alpha_i - \beta_i|x_2|, \forall i = 0, ..., n\}.$$

Note that since $\alpha_0 = 0$ and $\beta_0 = 1$, the polytope D_n is a subset of D from the previous example, but has suffered some cuts. The chamber complex of D_n has $(2n + 1)^2$ vertices that are not projections of vertices of the polytope itself. We depict D_2 in Figure 2.

Example 4.3 ($\mathscr{V}(D)$ can be exponentially larger than $\mathscr{K}(D)$)**.** Consider a polytope D_n in $\mathbb{R}^n \times \mathbb{R}$, where $n_x = n$ and $n_y = 1$, defined as the convex hull of the bottom hypercube $B = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\} \times \{0\}$



(a) The polytope D_2 from Example 4.2



(b) The chamber complex of D_2, with 25 interior vertices that are not projections of vertices of D_2

Figure 2: Polytope D_2 of Example 4.2 whose chamber complex has several vertices that are not projections of vertices of D_2

and the top hypercube $T = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1/2\} \times \{1\}$. The representation of D_n by linear inequalities can be written as

$$D_n = \left\{ (x, y) \ : \ \begin{array}{l} 0 \leq y \leq 1 \\ \frac{1}{2}y - 1 \leq x_i \leq 1 - \frac{1}{2}y, \ \forall i \in [n]. \end{array} \right\}$$

We call D_n the $(n+1)$-truncated Pyramid. Figure 3 illustrates the truncated pyramid D_2 and Figure 4 provides its chamber complex in $\mathbb{R}^2$. While it is no longer possible to draw D_3 as a 4-dimensional object, Figure 5 shows the chamber complex in this case.

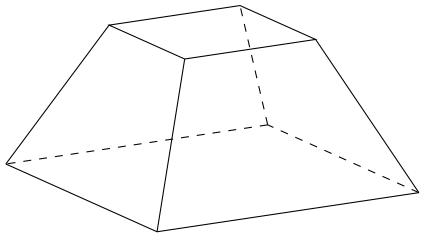


Figure 3: Truncated Pyramid with D_2

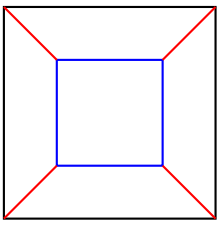


Figure 4: Chamber complex of D_2

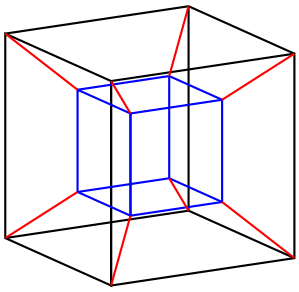


Figure 5: Chamber complex of D_3

The $(n+1)$-truncated pyramid D_n is an example where the number of vertices of the chamber complex is exponentially large ($|\mathscr{V}(D_n)| = 2^{n+1}$), while the number of full-dimensional chambers is relatively small ($|\mathscr{K}(D_n)| = 2n+1$). ◊

4.1 The Bilevel Continuous Knapsack Problem

Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n_+$ be a weight vector and let $A = \sum_{i=1}^n a_i$. The Bilevel Continuous Knapsack problem of a is given by

$$BK(a) := \left\{ \begin{array}{ll} \max\limits_{x} & \mathbb{E}[\langle d_2, y(x,\omega)\rangle] - \langle d_1, x\rangle \\ s.t. & 0 \leq x \leq A \\ & y(x,\omega) \text{ a.s. solves } \left\{ \begin{array}{ll} \max\limits_{y} & \langle c(\omega), y\rangle \\ s.t. & y \in [0,1]^n \\ & \sum_{i=1}^n a_i y_i \leq x, \end{array} \right. \end{array} \right. \tag{23}$$

where $d_1 \in \mathbb{R}_+$ is the cost of the knapsack capacity (decided by the leader), and $d_2 \in \mathbb{R}^n$ is the valorization of the leader over the objects 1 to n. Once the knapsack size is decided, the follower must choose how much of each (divisible) object to carry with, respecting the capacity constraint $\sum a_i y_i \leq x$, using as criteria the uncertain cost $c(\omega)$. A detailed study of this problem has been recently done in [5]. Furthermore, it is shown that the set of minimal chambers is given by

$$\mathscr{V}(D) = \left\{ \sum_{i \in I} a_i \ : \ I \subset [n] \right\}, \tag{24}$$

under the convention $\sum_{\emptyset} a_i = 0$. Admitting any values of $a \in \mathbb{R}^n_+$ it is possible to obtain different values for any partial sum in (24), leading to $|\mathscr{V}(D)| = 2^n$, which is exponentially large with respect to the $2n + 3$ constraints needed to define D. Furthermore, by ordering $\mathscr{V}(D)$, we obtain that $\mathscr{K}(D)$ is given by the induced intervals, and so $|\mathscr{K}(D)| = 2^n - 1$. Thus, this is an example where $\mathscr{K}(D)$ is almost as large as $\mathscr{V}(D)$, and both are exponentially large with respect to the data of the problem.
When $a \in \mathbb{N}^n$ (that is, the weights $(a_1, \ldots, a_n)$ are integers), one can give an overestimation of V given by $\mathscr{V}(D) \subset \{0\} \cup [A] = \{0, 1, 2, \ldots, A\}$. Thus, assuming that we have an oracle to evaluate the function $\theta(x) = \mathbb{E}[\langle d_2, y(x,\omega)\rangle] - \langle d_1, x\rangle$ for any $x \in [0, A]$ (or that we are solving a Sample approximation), then we can solve $BK(a)$ in pseudo-polynomial time by computing the values $\{\theta(0), \theta(1), \ldots, \theta(A)\}$ (the number A requires $\log(A)$ bits to be represented).

Remark 4.4. *In [5], it is shown that the evaluation of θ can be hard even for this problem, since solving $BK(a)$ is in general, #P-hard. Nevertheless, in the same work pseudo-polynomial algorithms are provided for some specific distributions of c.*

Example 4.5. Let us consider a very simple instance of BK, where $n = 2$, $a_1 = 1$, $a_2 = 2$, and c follows a uniform distribution over $\mathbb{S}_2$. In this case, the chamber vertices are exactly $\mathscr{V}(D) = \{0, 1, 2, 3\}$, and so, the full-dimensional chambers are given by $\mathscr{K}(D) = \{[0,1], [1,2], [2,3]\}$.

1. When $x \in [0,1]$: The set $S(x)$ is given as in Figure 7.(a) and the centroid of $S(x)$ is given by
$$\mathfrak{b}(x) = \left(\frac{1}{4} + \frac{\alpha}{2\pi}\right) \begin{bmatrix} 0 \\ x/2 \end{bmatrix} + \left(\frac{1}{4} + \frac{\pi/2 - \alpha}{2\pi}\right) \begin{bmatrix} x \\ 0 \end{bmatrix}$$
2. When $x \in [1,2]$: The set $S(x)$ is given as in Figure 7.(b) and the centroid of $S(x)$ is given by
$$\mathfrak{b}(x) = \left(\frac{1}{4} + \frac{\alpha}{2\pi}\right) \begin{bmatrix} 0 \\ x/2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{\pi/2 - \alpha}{2\pi} \begin{bmatrix} 1 \\ (x-1)/2 \end{bmatrix}$$
3. When $x \in [2,3]$: The set $S(x)$ is given as in Figure 7.(c) and the centroid of $S(x)$ is given by
$$\mathfrak{b}(x) = \frac{1}{4} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{\alpha}{2\pi} \begin{bmatrix} x-2 \\ 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{\pi/2 - \alpha}{2\pi} \begin{bmatrix} 1 \\ (x-1)/2 \end{bmatrix}$$

This computation allows to explicitly determine $\theta(x)$, simply by noting that $\theta(x) = \langle d_2, \mathfrak{b}(x)\rangle - \langle d_1, x\rangle$. For appropriated values of d_1 and d_2, it is possible to set the minimum of θ to be attained at any of the chamber vertices. $\diamondsuit$

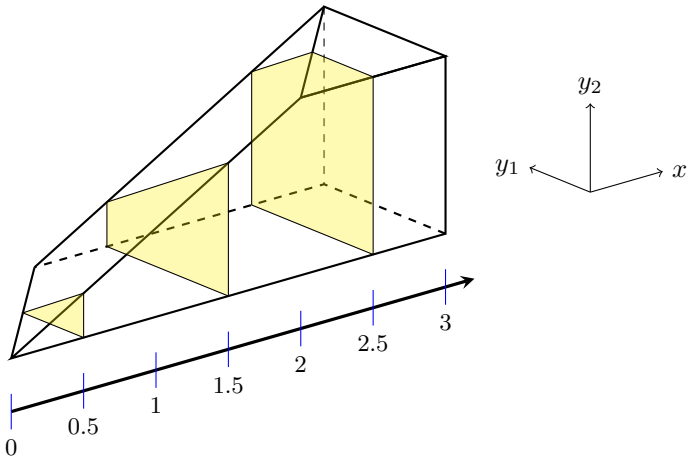
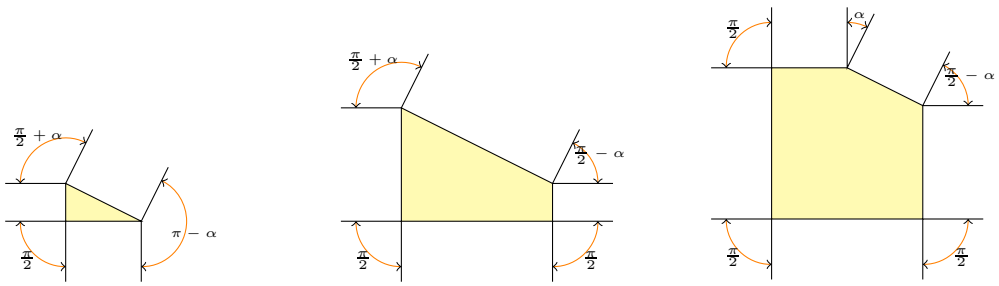


Figure 6: High-point relaxation of BK; In yellow, $S(0.5)$, $S(1.5)$ and $S(2.5)$.



(a) (b) (c)

Figure 7: From left to right: Slices $S(0.5)$, $S(1.5)$ and $S(2.5)$, including their normal fans. Here, $\alpha = \arcsin(1/\sqrt{5})$.

5 Enumeration algorithm

The rest of the work is focused on algorithms to solve Problem (14) in the cases when we can evaluate the objective function $x \mapsto \langle d_1, x\rangle + \mathbb{E}_{\beta_x}[\langle d_2, y\rangle]$, or to solve Problem (22), otherwise. The key observation is that, thanks to Theorems 3.6 and 3.8, both problems have the form

$$\min_{x \in X} \theta(x), \tag{25}$$

where $\theta : X \to \mathbb{R}$ is a continuous function, piecewise linear over the chamber complex $\mathscr{C}(D)$. Thus, we will provide algorithms to solve this generic problem.
In this setting, Corollary 3.2 gives us a natural strategy to solve Problem (25): It is enough to compute the chamber vertices $\mathscr{V}(D)$ and evaluate the corresponding objective function θ at each one of them. In this section we provide an enumeration algorithm to compute $\mathscr{V}(D)$ by sequentially solving mixed-integer programming problems which are formulated using the Face Lattice $\mathscr{F}$.

Remark 5.1. *Computing $\mathscr{V}(D)$ is at least as hard as computing all vertices of a polytope. Indeed, given an arbitrary (full-dimensional) polytope $P \subseteq \mathbb{R}^n$, one can consider*

$$D = \{(x, y) \in \mathbb{R}^{n+1} \,:\, x \in P,\, 0 \leq y \leq 1\} = P \times [0, 1].$$

It is not hard to see that $\mathscr{V}(D)$ corresponds exactly to the extreme points of P. This follows since all faces of D are either parallel to P or orthogonal to P. In both cases, the projection of faces D are also faces of P (see Figure 8).

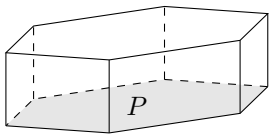


Figure 8: Illustration of D (the volume) as direct lifting of P (in gray).

To the best of our knowledge, the complexity of finding all vertices of a polytope P is currently unknown; however, for a polyhedron P (not necessarily bounded) it is known that it is NP-complete to decide, given a subset of vertices of P, if there is a new vertex of P to add to the collection [19].

5.1 Mixed-Integer formulation

To compute chambers, Definition 2.4 tells us that we need access to the Face Lattice $\mathscr{F}$. Proposition 2.6 improves this requirement reducing $\mathscr{F}$ only to

$$\mathscr{F}_{\leq n_x} := \{F \in \mathscr{F} \ : \dim(F) \leq n_x\} = \bigcup_{n=0}^{n_x} \mathscr{F}_n.$$

Recalling the representation (10), we have that each Face $F \in \mathscr{F}_n$ is represented by a set of constraints $J \subset [m]$ with $|J| = \operatorname{codim}(F) = (n_x + n_y) - n$. Thus,

$$|\mathscr{F}_{\leq n_x}| = \sum_{n=0}^{n_x} |\mathscr{F}_n| \leq \sum_{n=n_y}^{n_x+n_y} \binom{m}{n} = O(m^{n_x+n_y}). \tag{26}$$

If the dimensions n_x and n_y are fixed, the problem of "computing a chamber" can be solved as a Linear Programming problem of polynomial size, as the following proposition shows.

Proposition 5.2. *Every chamber $C \in \mathscr{C}(D)$ can be uniquely identified with the maximal subset $\mathcal{I} \subset \mathscr{F}_{\leq n_x}$ such that $C = \bigcap_{F\in\mathcal{I}} \pi(F)$. Moreover, for any $x \in \operatorname{ri}(C)$, we can compute $\mathcal{I}$ as*

$$\mathcal{I} = \{F \in \mathscr{F}_{\leq n_x} \ : \ C \subseteq \pi(F)\} = \{F \in \mathscr{F}_{\leq n_x} \ : \ x \in \pi(F)\}. \tag{27}$$

Moreover, for every $x \in X$ (fixed) we can consider the problem

$$\begin{cases} \max\limits_{(y_F)} & \sum_{F\in\mathscr{F}_{\leq n_x}} g_F(x, y_F) \\ s.t. & By_F \leq b - Ax, \quad \forall F \in \mathscr{F}_{\leq n_x}. \end{cases} \tag{28}$$

Then, for any solution $(y_F^* \ : \ F \in \mathscr{F}_{\leq n_x})$, one has that the maximal set associated to $\sigma(x)$ is given by

$$\mathcal{I} = \{F \in \mathscr{F}_{\leq n_x} \ : \ g_F(x, y_F^*) = 0\}. \tag{29}$$

Proof. Fix $C \in \mathscr{C}(D)$, choose any $x \in \operatorname{ri}(C)$, and define $\mathcal{I} = \{F \in \mathscr{F}_{\leq n_x} \ : \ x \in \pi(F)\}$. Recalling from Proposition 2.6 that $C = \sigma(x)$, by construction we get that

$$C = \sigma(x) = \bigcap_{F\in\mathcal{I}} \pi(F).$$

In particular, we deduce that $\mathcal{I} \subseteq \{F \in \mathscr{F}_{\leq n_x} \ : \ C \subseteq \pi(F)\}$ Suppose that there exists another set $\mathcal{I}' \subset \mathscr{F}_{\leq n_x}$ such that $C = \bigcap_{F\in\mathcal{I}'} \pi(F)$. Then, if $\mathcal{I}' \not\subset \mathcal{I}$ there would exists a face $F \in \mathcal{I}' \setminus \mathcal{I}$ and so, by

construction of $\mathcal{I}$ one would get that $x \notin \pi(F)$, which is a contradiction. Thus, $\mathcal{I}' \subset \mathcal{I}$, proving the maximality (and therefore, uniqueness) of $\mathcal{I}$. Finally, maximality also entails that

$$\mathcal{I} \supseteq \{F \in \mathscr{F}_{\leq n_x} \ : \ C \subseteq \pi(F)\}$$

finishing the first part of the proof. The second part of the proof follows directly recalling that for every $x \in X$ one has that $x \in \mathrm{ri}(\sigma(x))$ and that for every face $F \in \mathscr{F}_{\leq n_x}$

$$x \in \pi(F) \iff \exists y_F^* \in S(x) \text{ such that } g_F(x, y_F^*) = 0.$$

Then, any solution $(y_F^* \ : \ F \in \mathscr{F}_{\leq n_x})$ of (28) must satisfy that $g_F(x, y_F^*) = 0$ for all faces in $\ell(x)$, and only for those faces. □

In what follows, we call the set $\mathcal{I}$ given by the above proposition the *label* of C, and denote it by $\ell(C)$. Similarly, for every $x \in X$ we call $\ell(x) := \ell(\sigma(x))$ the *label* of x. Observe that labels as defined follow the next natural lemma, which is the baseline of our enumeration algorithm.

Lemma 5.3. *For every $C_1, C_2 \in \mathscr{C}(D)$ one has that*

$$C_1 \subseteq C_2 \iff \ell(C_1) \supseteq \ell(C_2).$$

Thus, the set $\{\ell(v) \ : \ v \in \mathscr{V}(D)\}$ corresponds to the maximal elements of $\{\ell(C) \ : \ C \in \mathscr{C}(D)\}$.

Proof. We only need to prove the equivalence, since the second part of the Lemma is a direct conclusion of it. Let $C_1, C_2 \in \mathscr{C}(D)$. Proposition 5.2 yields directly the necessity (that is, the reverse implication). Now, for the sufficiency, suppose that $C_1 \subseteq C_2$, and choose $x \in \mathrm{ri}(C_1)$. Then,

$$x \in C_2 = \bigcap_{F \in \ell(C_2)} \pi(F),$$

and so, $\ell(x) \supseteq \ell(C_2)$. The proof follows recalling that since $x \in \mathrm{ri}(C_1)$, then $\ell(C_1) = \ell(x)$. □

Using these results, we can generate an element of $\mathscr{V}(D)$ through a MILP formulation that aims at finding a maximal element of $\{\ell(C) \ : \ C \in \mathscr{C}(D)\}$. The following formulation achieves this when D is a polytope:

$$\begin{aligned}
\max_{z,x,y} \quad & \sum_{F \in \mathscr{F}_{\leq n_x}} z_F & & & \text{(30a)}\\
\text{s.t.} \quad & Ax + By_F \leq b & & \forall F \in \mathscr{F}_{\leq n_x} & \text{(30b)}\\
& Ax + By_F \geq b - M(1 - z_F) & & \forall F \in \mathscr{F}_{\leq n_x} & \text{(30c)}\\
& z_F \in \{0, 1\} & & \forall F \in \mathscr{F}_{\leq n_x} & \text{(30d)}
\end{aligned}$$

Here, y and z stand for the vectors $(y_F \ : \ F \in \mathscr{F}_{\leq n_x})$ and $(z_F \ : \ F \in \mathscr{F}_{\leq n_x})$, respectively, and M is a vector of m positive values such that $A_i x + B_i y - b_i \geq -M_i$, for all $(x, y) \in D$. This is well defined when D is a polytope, and can be easily computed using m linear programs. Formulation (30) is straightforward: it tries to "activate" as many faces as possible such that the intersection of their projection is non-empty. Let (z^*, x^*, y^*) be an optimal solution of (30). It is not hard to see that x^* is an element of $\mathscr{V}(D)$, thus, we can collect it and focus on generating a new element of $\mathscr{V}(D)$. Noting that $\ell(x^*) = \{F \in \mathscr{F}_{\leq n_x} \ : \ z_F^* = 1\}$, we see that such new element can be obtained by adding the following inequality to (30):

$$\sum_{F \in \mathscr{F}_{\leq n_x} \, : \, z_F^* = 0} z_F \geq 1. \tag{31}$$

Since (z^*, x^*, y^*) induced a maximal element of $\{\ell(C) \ : \ C \in \mathscr{C}(D)\}$, we can easily see that constraint (31) is removing only the single element x^* of $\mathscr{V}(D)$ from (30).

This procedure can be iterated until the optimization problem becomes infeasible; however, in order to avoid detecting infeasibility in our computational implementation, we add a new binary variable s that can relax (31) when needed. Under these considerations, we present the precise model we use: suppose we have partially generated a set $V \subseteq \mathscr{V}(D)$, we generate an element of $\mathscr{V}(D) \setminus V$ or determine $V = \mathscr{V}(D)$ using the following optimization problem:

$$\max_{z,s,x,y} \quad \sum_{F \in \mathscr{F}_{\leq n_x}} z_F \tag{32a}$$

$$\text{s.t.} \quad Ax + By_F \leq b \qquad \forall F \in \mathscr{F}_{\leq n_x} \tag{32b}$$

$$Ax + By_F \geq b - M(1 - z_F) \qquad \forall F \in \mathscr{F}_{\leq n_x} \tag{32c}$$

$$\sum_{F \not\in \ell(v)} z_F + s \geq 1 \qquad \forall v \in V \tag{32d}$$

$$\sum_{F \in \mathscr{F}_{\leq n_x}} z_F \leq |\mathscr{F}_{\leq n_x}|(1 - s) \tag{32e}$$

$$z_F \in \{0, 1\} \qquad \forall F \in \mathscr{F}_{\leq n_x} \tag{32f}$$

$$s \in \{0, 1\} \tag{32g}$$

Lemma 5.4. *Problem* (32) *is always feasible provided that* $D \neq \emptyset$*. Moreover, in an optimal solution* (z^*, s^*, x^*, y^*)*, then one (and only one) of the following situations hold*

- $s^* = 0$ *and* $x^* \in \mathscr{V}(D) \setminus V$*.*
- $s^* = 1$ *and* $\mathscr{V}(D) = V$*.*

Proof. Let us first show that (32) is feasible. Since D is nonempty, there exists at least one point $(\bar{x}, \bar{y}) \in D$. Then, we can consider $\hat{s} = 1$, $x = \bar{x}$, and, for each $F \in \mathscr{F}_{\leq n_x}$, $\hat{z}_F = 0$ and $\hat{y}_F = \bar{y}$. It is easy to verify that $(\hat{z}, \hat{s}, \hat{x}, \hat{y})$ is feasible.
For the second part of the statement, we begin by tackling the case of $s^* = 0$. By constraint (32d), we know that $x^* \notin V$ since, for each $v \in V$, $z^*_F = 1$ for some $F \not\in \ell(v)$ (recall that $\ell(v)$ is the *maximal* set of faces whose intersection of projections is $\{v\}$, see Proposition 5.2). The fact that $x^* \in \mathscr{V}(D)$ follows from the optimality of the solution and Lemma 5.3, as constraint (32d) only removes elements of V.
Now assume $s^* = 1$. Constraint (32e) implies that $z^*_F = 0$, $\forall F \in \mathscr{F}_{\leq n_x}$. If there were an element $\tilde{x} \in \mathscr{V}(D) \setminus V$, we could clearly set $\tilde{z}_F = 1$ $\forall F \in \ell(\tilde{x})$ and find $\tilde{y}_F$ according to (28). The vector $(\tilde{z}, 0, \tilde{x}, \tilde{y}_F)$ would be feasible and, since $\ell(\tilde{x})$ must be nonempty,

$$\sum_{F \in \mathscr{F}_{\leq n_x}} z^*_F = 0 < \sum_{F \in \mathscr{F}_{\leq n_x}} \tilde{z}_F$$

This contradicts the optimality of (z^*, s^*, x^*, y^*), thus $\mathscr{V}(D) = V$. □

In Algorithm 1 we formalize our enumeration procedure. The correctness of the Algorithm is given by Lemma 5.4.
To solve Problem (25), it is enough to run Algorithm 1, and evaluate θ over the set $V = \mathscr{V}(D)$.

5.2 Practical computational considerations

Algorithm 1 is basically made up of two (expensive) computations: the (partial) Face Lattice $\mathscr{F}_{\leq n_x}$ and solving Problem (32). To compute $\mathscr{F}_{\leq n_x}$ is as hard as computing the whole Face Lattice. Thus, for Face Lattice computations, we rely on Polymake [14], which is a highly optimized package for these purposes. For solving Problem (32) we use Gurobi [15], which we enhance using the following simple local search heuristic.

Algorithm 1: Chamber vertex enumeration algorithm

Input: A, B, d defining a polytope $D = \{(x, y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} : Ax + By \leq b\}$;
Set $V = \emptyset$, $s^* = 0$;
Compute $\mathscr{F}_{\leq n_x}(D)$;
while *true* **do**
  Solve Problem (32) and obtain an optimal solution (z^*, s^*, x^*, y^*);
  if $s^* = 0$ **then**
    $V \leftarrow V \cup \{x^*\}$;
    Store $\ell(x^*) = \{F \in \mathscr{F}_{\leq n_x}(D) : z^*_F = 1\}$;
  else
    break;
  end
end

Result: The set $V = \mathscr{V}(D)$

A local-search heuristic: When solving Problem (32) in Algorithm 1, it is not strictly necessary to have the true optimal solution. A feasible solution $(\hat{z}, \hat{s}, \hat{x}, \hat{y})$ such that $\hat{y}$ induces a maximal element of $\{\ell(C) : C \in \mathscr{C}(D)\}$ suffices. We can exploit this in the following way: whenever a feasible solution $(\hat{z}, \hat{s}, \hat{x}, \hat{y})$ is found by Gurobi, we flip any component $\hat{y}_F = 0$ to 1, and check if the x components can be modified to obtain a new feasible solution to (32). This is a linear program. We repeat the process for each $\hat{y}_F = 0$ and pass the resulting solution to Gurobi via a callback.

6 Monte-Carlo Approximation scheme

As we have seen, the enumeration algorithm of Section 5 has several drawbacks. First, it requires to compute (in practice) the whole Face Lattice of D, which might depend exponentially on the whole dimension $n_x + n_y$. And even with the Face Lattice at hand, computing all chamber vertices in $\mathscr{V}(D)$ can be hard. Even worse, as we have seen in Section 4, $\mathscr{V}(D)$ might be exponentially large.
Another approach, that we explore in this section, is to try to compute the collection of full-dimensional chambers $\mathscr{K}(D)$. If one has access to this family instead, one could find the solution of Problem (25) as follows:

1. For each $K \in \mathscr{K}(D)$, compute $d_K \in \mathbb{R}^{n_x}$ such that $\theta\big|_K = \langle d_K, \cdot \rangle + a_K$ (the additive constant a_k can be disregarded), and solve the linear problem $\min_{x \in K} \langle d_K, x \rangle$.
2. Noting that $X = \bigcup_{K \in \mathscr{K}(D)} K$, the solution x^* with minimal value $\theta(x^*)$ among those found in the first step must be the global solution of Problem (25).

While listing the whole family $\mathscr{K}(D)$ could be exponentially long, as the Knapsack problem illustrates in Section 4, we have an advantage: If we draw a point $x \in X$ randomly, $\sigma(x)$ will be a full-dimensional chamber almost surely.

6.1 Label representation for the full-dimensional chambers

To simplify the exposition, from now on, we will assume that $X \subset [0, 1]^{n_x}$ and we will write $X^c = [0, 1]^{n_x} \setminus X$. Since D is a polytope, this requirement can be easily attained by a mild change of variables. To be able to consider samples in $[0, 1]^{n_x}$ (which are easy to generate, and they are independent of the specific structure of D), we identify $\ell(X^c)$ with $\emptyset$, which correspond to the set of faces that are activated by a point $x \in X^c$. We start with the following direct lemma, that improves Proposition 2.6.

Lemma 6.1. *let $K \in \mathscr{K}(D)$. Then, for any $x \in \mathrm{int}(K)$, we have that*

$$K = \bigcap \{\pi(F) \ : \ x \in \pi(F),\, \dim(F) = n_x\}\,. \tag{33}$$

Proof. We only need to show that for every face $F \in \ell(K)$, $\dim(F) \geq n_x$, thanks to Proposition 2.6 and Proposition 5.2. But this is clear since the inclusion $K \subset \pi(F)$ entails that $n_x = \dim(K) \leq \dim(F)$. □

Observe that this is already a substantial improvement: to compute $\mathscr{F}_{n_x}$ is extremely less demanding than computing $\mathscr{F}_{\leq n_x}$. Moreover, the size of $\mathscr{F}_{n_x}$ can be controlled by fixing only the follower's dimension n_y since

$$|\mathscr{F}_{n_x}| \leq |\{J \subset [m] \ : \ |J| = n_y\}| = \binom{m}{n_y} = O(m^{n_y}). \tag{34}$$

The next lemma, which is also direct, allows us to compute the linear component of θ in a full-dimensional chamber K as its derivative at any point $x \in \mathrm{int}(K)$.

Lemma 6.2. *Let $\bar{x} \in X$ and let $\ell = \ell(\bar{x})$. If $\sigma(\bar{x})$ is a full-dimensional chamber, then for each $j \in [n_x]$, the following problem*

$$\begin{cases} \max\limits_{t,\,(y_F)_{F\in\ell}} & t \\ s.t. & (\bar{x} + te_j, y_F) \in D, \qquad \forall F \in \ell, \\ & g_F(\bar{x} + te_j, y_F) = 0, \qquad \forall F \in \ell, \end{cases} \tag{35}$$

has a solution $t_j^* > 0$. Moreover, for every function $\theta : X \to \mathbb{R}$ continuous and piecewise linear over the chamber complex $\mathscr{C}(D)$, the vector $d_\ell \in \mathbb{R}^{n_x}$ such that $\theta\big|_{K_\ell} = \langle d_\ell, \cdot\rangle + a_K$ (for some $a_K \in \mathbb{R}$) can be taken as

$$d_\ell = \begin{pmatrix} \frac{\theta(\bar{x}+t_1^* e_1)-\theta(\bar{x})}{t_1^*} \\ \vdots \\ \frac{\theta(\bar{x}+t_{n_x}^* e_{n_x})-\theta(\bar{x})}{t_{n_x}^*} \end{pmatrix}.$$

Proof. Let $K = \sigma(\bar{x})$. Since $\bar{x} \in \mathrm{ri}(K) = \mathrm{int}(K)$, it is clear that for every $j \in [n_x]$, the problem

$$\max_t \{t \ : \ \bar{x} + te_j \in K\}\,,$$

has a solution $t_j^* > 0$. The conclusion then follows by noting that the above optimization problem is equivalent to Problem (35). The rest is direct. □

Now, we have all the elements to establish our Monte-Carlo algorithm (see Algorithm 2) to approximate the solution of Problem 25, by randomly drawing points from $[0,1]^{n_x}$. Of course, as in Section 5, we need the function $\theta : X \to \mathbb{R}$ to be continuous, piecewise linear over $\mathscr{C}(D)$ and computable, in the sense that for any given $x \in X$, we can compute $\theta(x)$ efficiently.
The main drawback of the Algorithm 2 is that we don't know if the result $(\hat{x}, \hat{\theta})$ is an optimal solution of Problem 25 or not. We therefore need to provide a measure of how good the solution $(\hat{x}, \hat{\theta})$ might be.

6.2 Volume-Error estimators

We want to provide a good notion of error for the estimation $(\hat{x}, \hat{\theta})$ produced by Algorithm 2 of the solution of Problem (25). To do so, we propose to consider the volume of the "unseen chambers" during the execution of the Monte-Carlo algorithm.
Let then $\mathscr{L} = \mathcal{P}(\mathscr{F}_{n_x})$. Recalling the identification $\ell(X^c)$ with $\emptyset$, we get that the set $\{\ell(K) \ : \ K \in \mathscr{K}(D) \cup \{X^c\}\}$ is contained in $\mathscr{L}$. Now, for each $\ell \in \mathscr{L}$, we define

$$\begin{aligned} \mathbb{1}_\ell : [0,1]^{n_x} &\to \{0,1\} \\ x &\mapsto \begin{cases} 1 & \text{if } \ell(x) = \ell, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{36}$$

Algorithm 2: Monte-Carlo algorithm

Input: A, B, d defining a polytope $D = \{(x, y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} : Ax + By \leq d\}$, a function θ continuous, computable, and piecewise linear over $\mathscr{C}(D)$;
Generate a (uniformly iid) training sample S of size N over $[0,1]^{n_x}$;
Set `List1` $= \emptyset$,`List2` $= \emptyset$, $\hat{x} = NaN$, $\hat{\theta} = \infty$;
foreach $\xi \in S$ **do**
    Compute $\ell = \{F \in \mathscr{F}_{n_x} : \xi \in \pi(F)\}$;
    `List1` $\leftarrow$ `List1` $\cup \{(\xi, \ell)\}$;
    if $\ell \in$ *`List2` or* $\ell = \emptyset$ **then**
        continue;
    else
        `List2` $\leftarrow$ `List2` $\cup \{\ell\}$;
        Compute d_ℓ as in Lemma 6.2;
        Solve the problem

$$\begin{cases} \min\limits_{x,(y_F)_{F\in\ell}} & \langle d_\ell, x\rangle & \\ s.t. & Ax + By_F \leq b & \forall F \in \ell \\ & g_F(x, y_F) = 0 & \forall F \in \ell \end{cases}$$

        finding a solution $\hat{x}_\ell$ and set the value $\hat{\theta}_\ell = \theta(\hat{x}_\ell)$;
        if $\hat{\theta}_\ell < \hat{\theta}$ **then**
            $\hat{x} \leftarrow \hat{x}_\ell$, $\hat{\theta} \leftarrow \hat{\theta}_\ell$;
        end
    end
end

Result: The pair solution-value $(\hat{x}, \hat{\theta})$ for problem (25), and the lists `List1`, `List2`.

Clearly, $\mathbb{1}_\ell = \mathbb{1}_{\text{int}(K_\ell)}$ where

$$K_\ell = \begin{cases} K \in \mathscr{K}(D) & \text{if } \ell = \ell(K) \\ X^c & \text{if } \ell = \emptyset \\ \emptyset & \text{otherwise.} \end{cases} \tag{37}$$

In what follows, for a set $K \subset \mathbb{R}^{n_x}$ we write $\text{Vol}(K)$ as its n_x-dimensional volume: that is, $\text{Vol}(K) = \mathcal{L}^{n_x}(K)$, where $\mathcal{L}^{n_x}$ stands for the usual Lebesgue measure over $\mathbb{R}^{n_x}$.

Proposition 6.3. *Let $\{\ell_1, \ell_2, \ldots, \ell_p\}$ be a subset of $\mathscr{L}$ with no repeated elements. Let Z be a uniformly distributed random vector over $[0,1]^{n_x}$. Then,*

1. $\sum_{i=1}^p \mathbb{1}_{\ell_i}$ *coincides with the indicator function of* $\bigcup_{i=1}^p \text{int}(K_{\ell_i})$.
2. $\mathbb{E}\left[\sum_{i=1}^p \mathbb{1}_{\ell_i}(Z)\right] = \text{Vol}\left(\bigcup_{i=1}^p K_{\ell_i}\right) = \sum_{i=1}^p \text{Vol}\left(K_{\ell_i}\right)$.
3. $\text{Var}\left[\sum_{i=1}^p \mathbb{1}_{\ell_i}(Z)\right] \leq 1$.

Proof. We will prove each statement separately.

1. Since the labels $\{\ell_1, \ell_2, \ldots, \ell_p\}$ are all different, and noting that $\{\text{ri}(K) : K \in \mathscr{C}(D)\} \cup \{X^c\}$ is a partition of $[0,1]^{n_x}$, we get that the union $U = \bigcup_{i=1}^p \text{int}(K_{\ell_i})$ is pairwaise disjoint. Then,

$$\mathbb{1}_U(x) = 1 \iff \exists! i \in [p], x \in K_{\ell_i} \iff \sum_{i=1}^p \mathbb{1}_{\ell_i}(x) = 1.$$

Noting that $\ell(x)$ is unique for each $x \in [0,1]^{n_x}$, the function $\sum_{i=1}^{p} \mathbb{1}_{\ell_i}$ only takes values in $\{0,1\}$, which finishes the proof.

2. It follows directly from 1. by noting that the uniform distribution of Z entails that $\mathbb{E}[\mathbb{1}_O(Z)] = \mathrm{Vol}(O)$ for each $U \subset [0,1]^{n_x}$.

3. Setting $U = \bigcup_{i=1}^{p} \mathrm{int}(K_{\ell_i})$, one has that $\mathrm{Vol}(U) \leq 1$. Then $\max\{\mathrm{Vol}(U), 1 - \mathrm{Vol}(U)\}$ is less than 1. Finally, one can write

$$\begin{aligned}\mathrm{Var}\left[\sum_{i=1}^{p} \mathbb{1}_{\ell_i}(Z)\right] &= \int_{[0,1]^{n_x}} (\mathbb{1}_U(z) - \mathrm{Vol}(U))^2 dz \\ &\leq \int_{[0,1]^{n_x}} \max\{\mathrm{Vol}(U), 1 - \mathrm{Vol}(U)\}^2 dz \leq \int_{[0,1]^{n_x}} 1 dz = 1.\end{aligned}$$

The proof is then finished. □

Now, let us consider a sample $S = \{\xi_1, \ldots, \xi_N\}$ of independent uniformly distributed random variables over $[0,1]^{n_x}$. We define the random set $K(S)$ as

$$K(S) := \bigcup_{i=1}^{N_1} K_{\ell(\xi_i)}.$$

Let $\mathcal{N} = [0,1]^{n_x} \setminus \bigcup\{\mathrm{int}(K) \ : \ K \in \mathscr{K}(D) \cup X^c\}$. Noting that $\mathcal{L}^{n_x}(\mathcal{N}) = 0$, and using the fact that for each $x \in [0,1]^{n_x} \setminus \mathcal{N}$ there is one and only one $\ell \in \mathscr{L}$ such that $\mathbb{1}_\ell(x) = 1$, the complement of $K(S)$, namely $K(S)^c = [0,1]^{n_x} \setminus K(S)$, can be easily represented as its indicator function, which can be written (almost surely) as

$$\mathbb{1}_{K(S)^c}(x) = \sum_{\ell \in \mathscr{L}} \left[\prod_{i=1}^{N} (1 - \mathbb{1}_\ell(\xi_i))\right] \mathbb{1}_\ell(x), \quad \forall x \in [0,1]^{n_x} \setminus \mathcal{N}.$$

Definition 6.4. *For a sample S of $[0,1]^{n_x}$, we define the volume-error estimator as*

$$\rho(S) = \mathrm{Vol}(K(S)^c) = 1 - \mathrm{Vol}(K(S)^c). \tag{38}$$

The volume-error estimator can be interpreted as follows: For a given sample S and an optimal solution x^* of problem (25), $\rho(S)$ is the probability of $x^* \in K(S)^c$, if x^* were a random vector following a uniform distribution in $[0,1]^{n_x}$. Another interpretation is that $\rho(S)$ is the proportion of points that still have a chance to have a lower value than $\hat{\theta}$ since, by construction, $\hat{\theta} \leq \theta(x)$ for all $x \in K(S)$. While this is appealing as error measure when no supplementary information is extracted from Problem (25), it can not be computed using the whole sample S. Indeed, using the data in S the estimation of $\rho(S)$ would always be 0.
Thus, we will divide the sample S in two samples: a first sample $S_1 = \{\xi_1, \ldots, \xi_{N_1}\}$, called the *training sample*, and a second one $S_2 = \{\zeta_1, \ldots, \zeta_{N_2}\}$ of size N_2, called the *testing sample*. It is important to address that this division must be decided before any realization of $S = (S_1, S_2)$ is obtained, to preserve independence. With this division, we can define the estimator

$$U(S_1, S_2) = \frac{1}{N_2} \sum_{j=1}^{N_2} \mathbb{1}_{K(S_1)^c}(\zeta_j) = \frac{1}{N_2} \sum_{j=1}^{N_2} \sum_{\ell \in \mathscr{L}} \left[\prod_{i=1}^{N_1} (1 - \mathbb{1}_\ell(\xi_i))\right] \mathbb{1}_\ell(\zeta_j), \tag{39}$$

which selects a random set $K(S_1)$ given by all the chambers observed by the training sample S_1, and then estimates the volume of the complement using testing sample S_2. Observe that, from the array `List1`

produced by Algorithm 2, we can easily compute $U(S_1, S_2)$ as

$$U(S_1, S_2) = \frac{1}{N_2} \sum_{j=1}^{N_2} \left[\prod_{i=1}^{N_1} (1 - \mathbb{1}_{\ell(\zeta_j)}(\xi_i)) \right] = \frac{1}{N_2} \sum_{j=1}^{N_2} \left[\prod_{i=1}^{N_1} \big(\ell(\zeta_j) \neq \ell(\xi_i) \big) \right],$$

where $\big(\ell(\zeta_j) \neq \ell(\xi_i)\big)$ is interpreted as its associated boolean value (1 if the inequality holds, and 0 otherwise).

Proposition 6.5. *Let (S_1, S_2) be a fixed training-testing division for the sample S. The following assertions hold:*

- *The conditional expectation $\mathbb{E}[U(S_1, S_2)|S_1]$ is well-defined, S_1-measurable and it verifies that*

$$\mathbb{E}[U(S_1, S_2)|S_1] = \mathrm{Vol}(K(S_1)^c) = 1 - \mathrm{Vol}(K(S_1)).$$

- *The expected error is bounded by $N_2^{-1/2}$, that is,*

$$\mathbb{E}\Big[|e(S_1, S_2)|\Big] \leq \frac{1}{\sqrt{N_2}},$$

where $e(S_1, S_2) = \mathrm{Vol}(K(S_1)^c) - U(S_1, S_2)$.

Proof. Set $\Omega_1 = \big([0,1]^{n_x}\big)^{N_1}$ and $\Omega_2 = \big([0,1]^{n_x}\big)^{N_2}$. Let $\mathbb{P}_1$ be the Lebesgue probability measure over Ω_1 and $\mathbb{P}_2$ be the Lebesgue probability measure over Ω_2. By its construction in (39), it is clear that U is $(\mathbb{P}_1 \times \mathbb{P}_2)$-integrable in the probability product space $(\Omega_1, \mathcal{B}(\Omega_1), \mathbb{P}_1) \times (\Omega_2, \mathcal{B}(\Omega_2), \mathbb{P}_2)$, and thus, Fubini's theorem (see, e.g., [21, Theorem 14.19]) entails that

$$\mathbb{E}[U(S_1, S_2)|S_1] = \int U(S_1, S_2) d\mathbb{P}_2(S_2)$$

is well-defined and $\mathbb{P}_1$-integrable. Now, for a fixed value of S_1, we get $K(S_1)^c$ is a fixed closed set and that $U(S_1, S_2) = \frac{1}{N_2} \sum_{j=1}^{N_2} \mathbb{1}_{K(S_1)^c}(\zeta_j)$. Thus,

$$\int U(S_1, S_2) d\mathbb{P}(S_2) = \mathbb{E}_{S_2} \left[\frac{1}{N_2} \sum_{j=1}^{N_2} \mathbb{1}_{K(S_1)^c}(\zeta_j) \right] = \frac{1}{N_2} \sum_{j=1}^{N_2} \mathbb{E}_{\zeta_j}[\mathbb{1}_{K(S_1)^c}(\zeta_j)] = \mathrm{Vol}(K(S_1)^c).$$

This finishes the first part of the proof. Now, for the second part, we first observe that

$$e(S_1, S_2)^2 = (\mathbb{E}[U(S_1, S_2)|S_1] - U(S_1, S_2))^2, \tag{40}$$

which is $(\mathbb{P}_1 \times \mathbb{P}_2)$-integrable. Again, applying Fubini's theorem, we get that

$$\mathbb{E}[e(S_1, S_2)^2] = \mathbb{E}\Big[\mathbb{E}[e(S_1, S_2)^2|S_1]\Big] = \mathbb{E}\Big[\mathrm{Var}[U(S_1, S_2)|S_1]\Big].$$

Finally, the conditional variance $\mathrm{Var}[U(S_1, S_2)|S_1]$ is just given by the variance of the mean estimator of $\mathbb{1}_{K(S_1)^c}$ for the sample S_2, that is,

$$\begin{aligned} \mathrm{Var}[U(S_1, S_2)|S_1] &= U(S_1, S_2)^2 - \mathbb{E}[U(S_1, S_2)|S_1]^2 \\ &= \left(\frac{1}{N_2} \sum_{j=1}^{N_2} \mathbb{1}_{K(S_1)^c}(\zeta_j) \right)^2 - \mathrm{Vol}(K(S_1)^c)^2. \end{aligned}$$

Assuming $K(S_1)^c$ as fixed, the above expression is bounded by $1/N_2$ (see, e.g., [22, Chapter 1]), provided that $\mathrm{Var}(\mathbb{1}_{K(S)^c}) \leq 1$ thanks to Proposition 6.3. Thus, by a mild application of Jensen's inequality, we get that

$$\begin{aligned}\mathbb{E}[|e(S_1,S_2)|] &\leq \sqrt{\mathbb{E}[e(S_1,S_2)^2]} \\ &= \left(\int \mathrm{Var}[U(S_1,S_2)|S_1]d\mathbb{P}(S_1)\right)^{-1/2} \leq \left(\int \frac{1}{N_2}d\mathbb{P}(S_1)\right)^{-1/2} = \frac{1}{\sqrt{N_2}}.\end{aligned}$$

□

For a large enough training sample S_1, we would expect $\mathrm{Vol}(K(S_1)^c)$ to be small with high probability. However, we don't know how small $\mathrm{Vol}(K(S_1)^c)$ actually is. The size of $\mathrm{Vol}(K(S_1)^c)$ is estimated with S_2.

Theorem 6.6. *For a fixed training-testing division (S_1, S_2) of sample S and for a confidence level $\varepsilon > 0$, the volume-error estimator $\rho(S)$ is less than $U(S_1,S_2) + \frac{\varepsilon^{-1}}{\sqrt{N_2}}$, that is,*

$$\mathbb{P}\left[\rho(S) \leq U(S_1,S_2) + \frac{\varepsilon^{-1}}{\sqrt{N_2}}\right] \geq 1-\varepsilon. \tag{41}$$

Proof. We know that $\rho(S) = \mathrm{Vol}(K(S_1 \cup S_2)^c) \leq \mathrm{Vol}(K(S_1)^c)$. Thus, as a mild application of Markov's inequality, we can write

$$\begin{aligned}\mathbb{P}\left[\rho(S) \leq U(S_1,S_2) + \frac{\varepsilon^{-1}}{\sqrt{N_2}}\right] &\geq \mathbb{P}\left[\mathrm{Vol}(K(S_1)^c) \leq U(S_1,S_2) + \frac{\varepsilon^{-1}}{\sqrt{N_2}}\right] \\ &\geq \mathbb{P}\left[\left|\mathrm{Vol}(K(S_1)^c) - U(S_1,S_2)\right| \leq \frac{\varepsilon^{-1}}{\sqrt{N_2}}\right] \\ &= 1 - \mathbb{P}\left[\left|e(S_1,S_2)\right| > \frac{\varepsilon^{-1}}{\sqrt{N_2}}\right] \\ &\geq 1 - \frac{\mathbb{E}[|e(S_1,S_2)|]}{\varepsilon^{-1}/\sqrt{N_2}}\end{aligned}$$

Since $\mathbb{E}[|e(S_1,S_2)|] \leq 1/\sqrt{N_2}$ by Proposition 6.5, we deduce (41), finishing the proof. □

7 Numerical Experiments

The goal of this section is to illustrate how both Algorithms work in some instances of Bilevel optimization with uncertain cost. To do so, we adapt some deterministic bilevel problems available in the literature.

7.1 Set-up and instances

To test our two approaches, we implemented both Algorithms 1 and 2 in Julia 1.8.2 [4], using Polymake [14] to compute face lattices and Gurobi 9.5.2 [15] to solve (32) and any auxiliary LP. All experiments were run on a Linux machine with an Intel Xeon Silver 4210 2.2G CPU and 128 GB RAM. All experiments were run using a single thread. The main objectives behind these experiments are (1) to determine how Algorithm 1 scales and (2) how well the Monte-Carlo algorithm performs in comparison to the exact method. A global time limit of 15 minutes was set for Algorithm 1; in case this time is met, only the chamber vertices that were found are used.

We use instances from two publicly available libraries: BOLib [30] and the bilevel instances in [1], which we call CoralLib. Since our approach relies on computing a face lattice, we can only consider low-dimensional instances at the moment: we restrict to $n_x + n_y \leq 10$. We consider the follower's cost in (1) to be uniformly distributed in the unit n_y-dimensional sphere.

Additionally, we consider randomly generated instances of the Stochastic Bilevel Continuous Knapsack Problem [5]. While these instances have a more efficient algorithm for them than the one presented here, they are helpful in showing how well our general-purpose stochastic algorithm performs. These instances have the following form:

$$\begin{aligned} \max_{x} \quad & -\delta x + d^\top y \\ s.t. \quad & x \in [L, U] \\ & y(x,\omega) \text{ solves } \begin{cases} \min\limits_{y} & \langle c(\omega), y\rangle \\ s.t. & a^\top y \le x, \qquad \text{a.s. } \omega \in \Omega, \\ & y \in [0,1]^{n_y} \end{cases} \end{aligned} \tag{42}$$

In our experiments, we consider a to be a random non-negative vector, $\delta = 1/4$ and d the vector of ones. We call *Knapsack_i* an instance generated for $n_y = i$.

In all experiments, we used a sample of size 100 for the follower's cost vector. The same sample is used in both algorithms to better compare their performance. Additionally, in Algorithm 2 we used a training sample of size 100 and a testing sample of size 100 as well.

In Tables 1, 2 and 3, we compare the performance of both methods. The "Obj gap" columns shows the gap between the values found for both algorithms; a negative gap indicates the stochastic method performed better. The "Error" column shows the error estimation as per (39). All execution times are in seconds.

Instance	Dim	Obj gap	Error	Time Alg. 1	Time Alg. 2
AnandalinghamWhite1990	2	0.0%	0.0%	3.4	6.8
Bard1984a	2	0.0%	0.0%	3.5	6.6
Bard1984b	2	0.0%	0.0%	3.4	6.6
Bard1991Ex2	3	0.0%	0.0%	3.4	6.8
BardFalk1982Ex2	4	0.0%	54.0%	3.5	6.5
BenAyedBlair1990a	3	0.0%	0.0%	3.4	7.0
BenAyedBlair1990b	2	0.0%	0.0%	3.4	6.7
BialasKarwan1984a	3	0.0%	0.0%	3.5	7.0
BialasKarwan1984b	2	0.0%	0.0%	3.4	6.7
CandlerTownsley1982	5	4.6%	36.0%	6.6	15.2
ClarkWesterberg1988	2	0.0%	0.0%	3.4	6.6
ClarkWesterberg1990b	3	0.0%	0.0%	3.4	6.9
GlackinEtal2009	3	0.0%	46.0%	3.4	6.2
HaurieSavardWhite1990	2	0.0%	0.0%	3.4	6.7
HuHuangZhang2009	3	0.0%	0.0%	3.4	6.9
LanWenShihLee2007	2	0.0%	0.0%	3.4	6.8
LiuHart1994	2	0.0%	0.0%	3.4	6.6
MershaDempe2006Ex1	2	0.0%	0.0%	3.4	6.6
MershaDempe2006Ex2	2	0.0%	65.0%	3.4	5.4
TuyEtal1993	4	0.0%	45.0%	3.6	6.5
TuyEtal1994	4	0.0%	53.0%	3.6	6.8
VisweswaranEtal1996	2	0.0%	0.0%	3.4	6.6
WangJiaoLi2005	3	0.0%	0.0%	3.4	6.9

Table 1: Summary of results for Algorithms 1 and 2 for selected BOLib2 instances[30].

Instance	Dim	Obj gap	Error	Time Alg. 1	Time Alg. 2
linderoth	6	0.0%	85.0%	132.1	11.0
moore90	2	0.0%	0.0%	3.3	6.5
moore90_2	2	0.0%	0.0%	3.4	6.6

Table 2: Summary of results for Algorithms 1 and 2 for selected CoralLib instances [1]

Instance	Dim	Obj gap	Error	Time Alg. 1	Time Alg. 2
Knapsack_6	7	0.0%	0.0%	126.6	19.0
Knapsack_7	8	0.0%	0.0%	691.9	39.5
Knapsack_8	9	-73.6%	0.0%	919.4	89.5
Knapsack_9	10	-3600%	0.0%	907.1	213.6

Table 3: Summary of results for Algorithms 1 and 2 for Knapsack instances [5]

The gap measures how far the value of the stochastic algorithm is from the exact method, i.e., if val_i is the value obtained by Algorithm i, then the gap is

$$\text{Gap} = \frac{\text{val}_2 - \text{val}_1}{|\text{val}_1|}$$

Ideally, if given enough time, $\text{val}_1 \leq \text{val}_2$ and thus $\text{Gap} \geq 0$. However, since we ran Algorithm 1 with a time limit, it may be that $\text{Gap} < 0$, which indicates the Monte-Carlo algorithm performing better than the exact method.

7.2 Discussion of the results

The first impression using BoLib2 results in Table 1 is that both algorithms are equally competitive in small instances. Except for one of the instances, the optimality Gap is 0%, which implies that the Monte-Carlo algorithm in fact finds the optimal value. The same happens with CoralLib instances in Table 2. Even more, for one of the instances in CoralLib (*linderoth*), the Monte-Carlo algorithm attains the optimum considerably faster than the Enumeration algorithm. This is natural since the Enumeration algorithm scales very poorly.

However, this analysis for the Monte-Carlo algorithm is only possible due to the fact that we have the exact solution at hand to compare with. In practice, if the Enumeration algorithm can't be executed, we would need to rely on the volume-errors. in this line, the results are less positive. While most instances of BOLib2 and CoralLib have a volume-error of 0%, some instances have a very large error, even attaining 85% in the worst one, which happens to be the best one in time (*linderoth*). Probably, these instances have many small full-dimensional chambers, and so the volume-error estimator doesn't reflect how well the algorithm is performing (it largely overestimates the error).

The Kanpsack instances in Table 3 provide very insightful results. First, the Monte-Carlo algorithm performed substantially better that the Enumeration counterpart in terms of time. Furthermore, due to the time limit, the Monte-Carlo algorithm found a better solution for the most exigent instances. While in theory the Knapsack problem should have a large volume error (since it has as much full-dimensional chambers as chamber vertices, as illustrated in Section 4), the small error tells us that most of these chambers are, in fact, small. This could be exploited by a more adapted sample technique (instead of just plain Monte-Carlo).

It is important to note that, due to the small sizes of the training and testing samples, the 0% error in fact provides a large confidence interval. Indeed, using Theorem 6.6 with $\varepsilon = 172$, for those instances with zero error we can only guarantee that the volume of unseen chambers is less than $0.2 = 2/\sqrt{100}$ with probability 1/2.

All these results show that Monte-Carlo algorithms seem to be very promising and further more complex techniques should be explored.

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